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THE ASYMPTOTIC BEHAVIOR OF A SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS.*

By NORMAN LEVINSON.

---

1. We shall prove the following result.

THEOREM I. *Let the linear system of differential equations with constant coefficients*

$$y'_j(t) = \sum_{k=1}^{n} a_{jk} y_k(t) \qquad (j = 1, 2, \cdots, n), \tag{1}$$

have all of its solutions bounded as $t \to +\infty$. *If the coefficients* $f_{jk}(t)$ *of the linear system*

$$x'_j(t) = \sum_{k=1}^{n} f_{jk}(t) x_k(t), \qquad (j = 1, 2, \cdots, n), \tag{2}$$

are continuous and satisfy

$$\int^{\infty} | f_{jk}(t) - a_{jk} | \, dt < \infty, \qquad (j, k = 1, 2, \cdots, n), \tag{3}$$

then corresponding to any solution $x_j(t)$, $(j = 1, 2, \cdots, n)$, *of* (2) *there is a solution* $y_j(t)$, $(j = 1, \cdots, n)$, *of the simple system* (1) *containing sinusoidal terms only, such that* $x_j(t) - y_j(t) \to 0$ *as* $t \to +\infty$.

Theorem I generalizes a recent result of Wintner [4]. Wintner's result contains the added restriction that all the characteristic numbers of (1) must be purely imaginary, i.e. that the general solution of (1) consists of sums of pure sinusoids.

Here the general solution of (1) is of the form $y_j = \sum_{k=1}^{n} C_k \phi_{jk}$ where the C_k are arbitrary constants. Moreover there is an integer m, $0 \leqq m \leqq n$, such that

$$\phi_{jk} = A_{jk} e^{i\lambda_k t}, \qquad (k = 1, 2, \cdots, m;\ j = 1, 2, \cdots, n), \tag{4}$$

where the λ_k are real and the A_{jk} are constants. The terms of (4) are all sinusoids. The remaining terms in the solution of (1) are of the form

$$\phi_{jk} = P_{jk}(t) e^{(-\beta_k + i\lambda_k)t}, \qquad (k = m+1, \cdots, n;\ j = 1, 2, \cdots, n), \tag{5}$$

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* Received October 1, 1945.

where $\beta_k > 0$, and the $P_{jk}(t)$ are polynomials in t.

2. We shall assume in what follows that

$$\lambda_k \neq 0, \qquad (k = 1, 2, \cdots, m). \tag{6}$$

This is no real restriction. For if (6) is not satisfied the transformations $Y_j(t) = y_j(t)e^{-i\lambda t}$ and $X_j(t) = x_j(t)e^{-i\lambda_k t}$ with real $\lambda \neq \lambda_k$, $(k = 1, 2, \cdots, m)$, will transform (1) and (2) into new systems satisfying the hypothesis of Theorem I and such that for the new system (6) is satisfied. Proving Theorem I for the new systems will obviously yield the result for the original systems immediately.

We shall require

LEMMA 1. *The general solution of the system*

$$y'_j(t) = \sum_{k=1}^{n} a_{jk}y_k(t) + F_j(t), \qquad (j = 1, 2, \cdots, n),$$

can be written as

$$y_j = \sum_{k=1}^{m} \phi_{jk}(t) \int_{t_0}^{t} e^{-i\lambda_k \tau} \sum_{h=1}^{n} B_{kh}F_h(\tau)\,d\tau + \sum_{h=1}^{n} \int_{t_0}^{t} G_{jh}(t-\tau)F_h(\tau)\,d\tau + \sum_{k=1}^{n} C_k\phi_{jk}(t)$$

where the B_{kh} are constants, the C_k are arbitrary constants, and the $G_{jh}(t)$ satisfy

$$|G_{jh}(t)| \leqq Me^{-\beta t}, \qquad t \geqq 0, \tag{7}$$

where $\beta > 0$ and M are constants independent of j and h.

The proof of Lemma 1 will be given at the end of this paper.

Using the notation $f_{jk}(t) - a_{jk} = g_{jk}(t)$, (2) can be written as

$$x'_j(t) = \sum_{k=1}^{n} a_{jk}x_k + F_j \tag{8}$$

where

$$F_j = \sum_{k=1}^{n} g_{jk}(t)x_k(t). \tag{9}$$

Moreover from (3) we have

$$\int^{\infty} |g_{jk}(t)|\,dt < \infty. \tag{10}$$

To (8) we can apply Lemma 1 obtaining by also making use of (9),

$$(11)\quad x_j(t) = \sum_{k=1}^{m} \phi_{jk}(t) \sum_{h,s=1}^{n} B_{kh} \int_{t_0}^{t} e^{-i\lambda_k \tau} g_{hs}(\tau) x_s(\tau)\, d\tau + \sum_{h,s=1}^{n} \int_{t_0}^{t} G_{jh}(t-\tau) g_{hs}(\tau) x_s(\tau)\, d\tau + \sum_{k=1}^{n} C_k \phi_{jk}(t).$$

By proper choice of C_k any solution of (2) can be made to satisfy (11).

3. We shall now prove that the $x_j(t)$ are bounded as $t \to +\infty$. For the case where an n-th order linear differential equation with constants replaces (1) and an n-th order linear differential equation replaces (2) with a condition of the type (3) satisfied, Cesari [2] showed that the boundedness of the solutions of the simple system as $t \to +\infty$ implied that of the solutions of the perturbed system. A simplified proof of this has been given by Bellman [1]. That (3) is a "best possible" condition even if only boundedness is desired for the perturbed system follows from [3].

Using (4) and (7) we have from (11) that there exist an absolute constant M_1 and a constant $B > 0$ depending on the arbitrary constants C_k, such that

$$(12)\quad |x_j(t)| \leqq M_1 \sum_{h,s=1}^{n} \int_{t_0}^{t} |g_{hs}(\tau) x_s(\tau)|\, d\tau + B.$$

Setting

$$(13)\quad \sum_{h,s=1}^{n} |g_{hs}(\tau)| = H(\tau)$$

and

$$\max_{j=1,\ldots,n} |x_j(t)| = X(t)$$

we have from (12)

$$(14)\quad X(t) \leqq M_1 \int_{t_0}^{t} H(\tau) X(\tau)\, d\tau + B.$$

We also note that (10) implies

$$\int^{\infty} H(\tau)\, d\tau < \infty.$$

Clearly then we can choose t_0 so large that

$$(15)\quad M_1 \int_{t_0}^{\infty} H(\tau)\, d\tau < \tfrac{1}{2}.$$

(Of course the choice of t_0 may affect the magnitude of B but not of M_1.)

Now suppose that for some $t \geqq t_0$, $X(t) = 2B$. Then taking the smallest value of $t \geqq t_0$ for which this is true we have from (14) and (15)

$$2B < \tfrac{1}{2}(2B) + B = 2B$$

which is impossible. Thus $X(t) \leqq 2B$ and therefore

$$|x_j(t)| \leqq 2B, \qquad t \geqq t_0. \tag{16}$$

4. We now return to (11) and write

$$\int_{t_0}^{t} e^{-i\lambda_k \tau} g_{hs}(\tau) x_s(\tau)\, d\tau = \left(\int_{t_0}^{\infty} - \int_{t}^{\infty}\right) e^{-i\lambda_k \tau} g_{hs}(\tau) x_s(\tau)\, d\tau.$$

The first integral on the right is a constant. Modifying C_k, $(k = 1, 2, \cdots, m)$, to include the constants arising from these definite integrals we can write (11) as

$$x_j(t) = \sum_{k=1}^{m} C_k \phi_{jk}(t) + Q_j(t) + R_j(t) + S_j(t) \tag{17}$$

where

$$Q_j(t) = -\sum_{k=1}^{m} \phi_{jk}(t) \sum_{h,s=1}^{n} B_{kh} \int_{t}^{\infty} e^{-i\lambda_k t} g_{hs}(\tau) x_s(\tau)\, d\tau,$$

$$R_j(t) = \sum_{h,s=1}^{n} \int_{t_0}^{t} G_{jh}(t-\tau) g_{hs}(\tau) x_s(\tau)\, d\tau$$

$$S_j(t) = \sum_{k=m+1}^{n} C_k \phi_{jk}(t).$$

Using (4), (13) and (16) it follows that there exists a constant M_2 such that

$$|Q_j(t)| \leqq M_2 B \int_{t}^{\infty} H(\tau)\, d\tau.$$

Thus $Q_j(t)$ converges to zero as $t \to +\infty$.

Again, using (7) we have

$$|R_j(t)| \leqq 2BM \int_{t_0}^{t} e^{-\beta(t-\tau)} H(\tau)\, d\tau \leqq 2BM\, e^{-\frac{1}{2}\beta t} \int_{t_0}^{\frac{1}{2}t} H(\tau)\, d\tau + 2BM \int_{\frac{1}{2}t}^{t} H(\tau)\, d\tau.$$

Thus $R_j(t)$ converges to zero as $t \to +\infty$.

Finally by (5), $S_j(t)$ approaches zero as $t \to +\infty$. Thus (17) yields Theorem I.

5. We turn now to the proof of Lemma 1. This lemma results from representing the solution of

$$y'_j(t) = \sum_{k=1}^{n} a_{jk} y_k(t) + F_j(t), \qquad (j = 1, \cdots, n), \tag{18}$$

by what in operational calculus is called the superposition theorem.

We consider first the system

$$\psi'_{jh}(t) = \sum_{k=1}^{n} a_{jk}\psi_{kh}(t) + \delta_{jh}, \qquad (j = 1, 2, \cdots, n), \tag{19}$$

where h is an integer, $1 \leqq h \leqq n$, and δ_{jh} is 1 if $j = h$ and is zero otherwise. A particular solution of (19) is given by $\psi_{jh} = d_{jh}$ where the d_{jh} are constants satisfying the equations

$$0 = \sum_{k=1}^{n} a_{jk}d_{kh} + \delta_{jh}, \qquad (j = 1, 2, \cdots, n).$$

The possibility of satisfying these equations results immediately from the fact that the determinant of $(a_{jk}) \neq 0$ which, in turn, is a consequence of (6). The general solution of (19) is given by

$$\psi_{jh}(t) = \sum_{k=1}^{n} b_{kh}\phi_{jk}(t) + d_{jh}, \qquad (j = 1, 2, \cdots, n), \tag{20}$$

where the b_{kh} are constants. We choose the b_{kh} so that the $\psi_{jh}(t)$ satisfy the initial conditions

$$\psi_{jh}(0) = 0, \qquad (j = 1, 2, \cdots, n). \tag{21}$$

We shall now show that a solution of the system (18) is given by

$$y_j = \sum_{h=1}^{n} \int_{t_0}^{t} \psi'_{jh}(t - \tau)F_h(\tau)\,d\tau. \tag{22}$$

We have, on using (19) in (22),

$$y_j = \sum_{h,k=1}^{n} a_{jk} \int_{t_0}^{t} \psi_{kh}(t - \tau)F_h(\tau)\,d\tau + \int_{t_0}^{t} F_j(\tau)\,d\tau.$$

Differentiating and using (21) we have

$$y'_j = \sum_{h,k=1}^{n} a_{jk} \int_{t_0}^{t} \psi'_{kh}(t - \tau)F_h(\tau)\,d\tau + F_j(t).$$

Using (22) this becomes

$$y'_j = \sum_{k=1}^{n} a_{jk}y_k + F_j(t)$$

which is (18). Thus (22) is a particular solution of (18).

Using (20) and (4) we see that

$$\psi'_{jh} = \sum_{k=1}^{m} i\lambda_k b_k \phi_{jk}(t) + \sum_{k=m+1}^{n} b_{kh}\phi'_{jk}(t)$$

Recalling (5) and setting $\beta = \frac{1}{2} \min_{k>m} \beta_k$ we see that

$$\sum_{k=m+1}^{n} b_{kh}\phi'_{jk}(t) = G_{jh}(t)$$

where $G_{jh}(t)$ satisfies (7). Setting $i\lambda_k b_{kh} = B_{kh}$ we now have

$$\psi'_{jh} = \sum_{k=1}^{m} B_{kh}\phi_{jk}(t) + G_{jh}(t).$$

Using this in (22) we have, as the general solution of (18),

$$y_j = \sum_{k=1}^{m}\sum_{h=1}^{n} B_{kh} \int_{t_0}^{t} \phi_{jk}(t-\tau) F_h(\tau)\, d\tau + \sum_{h=1}^{n} \int_{t_0}^{t} G_{jh}(t-\tau) F_h(\tau)\, d\tau + \sum_{k=1}^{n} C_k \phi_{jk}(t).$$

Using (4) this yields the result of Lemma 1.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

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COMMENT ON THE PRECEDING PAPER.*

By HERMANN WEYL.

The following lines, though adding little to the substance of Professor Levinson's paper, may help to shed some light on his interesting result. This result is not limited to linear equations; it holds for such perturbations as are majorized by linear perturbations. Levinson passes from a given solution $\mathfrak{x}$ of the complete equation to a corresponding one $\mathfrak{z}$ of the approximate equation, $\mathfrak{x} \to \mathfrak{z}$. We add the inverse process $\mathfrak{z} \to \mathfrak{x}$: transition from the unperturbed to the perturbed phenomenon.

A system of differential equations for n functions $x_i(t)$ of the real variable t may be looked upon as a single differential equation for the vector (column) $\mathfrak{x}$ with the components x_i, and assuming the right member to consist of a linear part $A\mathfrak{x}$ with a constant coefficient matrix $A = \| a_{ik} \|$ and a perturbation $\mathfrak{v}$, we may write the equation in the form

$$d\mathfrak{x}/dt = A\mathfrak{x} + \mathfrak{v}, \qquad \mathfrak{v}(t) = \mathfrak{f}(t, \mathfrak{x}(t)), \tag{1}$$

$\mathfrak{f}(t, \mathfrak{x})$ being a given vector function of t and a variable vector $\mathfrak{x}$. Form $U(t) = e^{At}$, the one-parameter group of linear transformations generated by the infinitesimal A:

$$dU/dt = AU = UA, \quad U(0) = \text{unit matrix } E; \quad U(t-\tau) = U(t)U^{-1}(\tau),$$

and define the absolute values $\| \mathfrak{x} \|$, $\| A \|$ of vectors and matrices by

$$\| \mathfrak{x} \|^2 = \Sigma \, | x_i |^2, \qquad \| A \|^2 = \Sigma \, | a_{ik} |^2.$$

The equation (1) may be written

$$(d/dt)\,(U^{-1}\mathfrak{x}) = U^{-1}\mathfrak{v}$$

("variation of constants"), whence follows

$$\mathfrak{x}(t) = \mathfrak{z}(t) + U(t) \int_0^t U^{-1}(\tau)\mathfrak{v}(\tau)\,d\tau, \tag{2}$$

$$\mathfrak{z}(t) = U(t)\mathfrak{a} \qquad (\mathfrak{a} = \text{const.}) \tag{3}$$

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* Received December 5, 1945.

being a solution of

(4) $$d\mathfrak{z}/dt = A\mathfrak{z}.$$

It can also be verified directly that if $\mathfrak{x}$ is a solution of (1), then the $\mathfrak{z}$ defined by (2) or

(5) $$\mathfrak{z}(t) = \mathfrak{x}(t) - U(t)\int_0^t U^{-1}(\tau)\mathfrak{v}(\tau)d\tau = \mathfrak{x}(t) - \int_0^t U(t-\tau)\mathfrak{v}(\tau)d\tau$$

is a solution of $d\mathfrak{z}/dt = A\mathfrak{z}$. This is, in general form, the lemma from which Mr. Levinson's investigation starts.

Let us now assume that

(I) $$\| \mathfrak{f}(t,\mathfrak{x}) \| \leq \| \mathfrak{x} \| \cdot g(t).$$

If the perturbation is linear, $\mathfrak{f}(t,\mathfrak{x}) = G(t)\mathfrak{x}$, then (I) holds with $g(t) = \| G(t) \|$.

LEMMA. *Suppose*

$$\| \mathfrak{z}(t) \| \leq a, \qquad \| U(t) \| \leq c \quad \textit{for} \quad 0 \leq t \leq t_1$$

and set

$$c\int_0^t g(\tau)d\tau = h(t).$$

Then the equation (5) *implies*

$$\| \mathfrak{x}(t) \| \leq a \cdot e^{h(t)} \qquad (0 \leq t \leq t_1).$$

Proof. For $x(t) = \| \mathfrak{x}(t) \|$ one obtains, as a consequence of assumption (I), the integral inequality

$$x(t) \leq a + c\int_0^t x(\tau)g(\tau)d\tau = a + \int_0^t x(\tau)\cdot dh(\tau);$$

hence for $y(t) = x(t) - a\cdot e^{h(t)}$ the inequality

$$y(t) \leq \int_0^t y(\tau)\cdot dh(\tau).$$

Choose a constant b such that $y(t) \leq b$ for $0 \leq t \leq t_1$. The inequalities

$$y(t) \leq b\cdot (h(t))^n/n! \qquad (n = 0, 1, 2, \cdots;\ 0 \leq t \leq t_1)$$

then follow one after the other by induction. In the limit for $n \to \infty$ one gets $y(t) \leq 0$.

Let us now introduce Levinson's two hypotheses:

(II) convergence of $q = \int_0^\infty g(\tau)\,d\tau$;

(III) boundedness of $U(t)$ for $t \to \infty$, $\| U(t) \| \leq c$ for $t \geq 0$.

Then of course every solution $\mathfrak{z}$, (3), of (4) is bounded, $\| \mathfrak{z}(t) \| \leq a = c \| \mathfrak{a} \|$. Our lemma yields at once

$$\| \mathfrak{x}(t) \| \leq a^* = a \cdot e^{cq} \quad (\text{for } t \geq 0), \tag{6}$$

and hence the

THEOREM. *Under the hypotheses* (I), (II), (III) *every solution of* (1) *is bounded for* $t \geq 0$.

Instead of (6) Levinson uses the rougher estimate

$$\| \mathfrak{x}(t) \| \leq a/(1-\theta) \qquad (0 \leq t \leq t_1;\ \theta = h(t_1)),$$

which he establishes in a more indirect way. Its validity is limited to intervals $0t_1$ for which $\theta < 1$. Nonetheless, the theorem can be derived from it: one simply replaces the lower limit of integration 0 by a t_0 so chosen that

$$\theta = c \int_{t_0}^\infty g(\tau)\,d\tau < 1.$$

In a suitable coordinate system $U(t)$ breaks up into blocks (elementary divisors) of the form

$$\begin{Vmatrix} e^{\lambda t} \cdot t_0, & 0, & \cdot\ \cdot\ \cdot & 0 \\ e^{\lambda t} \cdot t_1, & e^{\lambda t} \cdot t_0, & \cdot\ \cdot\ \cdot & 0 \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \\ e^{\lambda t} \cdot t_{m-1}, & e^{\lambda t} \cdot t_{m-2}, & \cdot\ \cdot\ \cdot & e^{\lambda t} \cdot t_0 \end{Vmatrix} \qquad [\ t_i = (1/i!)t^i\ ].$$

Hence hypothesis (III) implies that either $\Re\lambda < 0$ or $\Re\lambda = 0$; but in the latter case the block must consist of one row only ($m = 1$). After uniting all elementary divisors of the first and second kind into V_1, V_2 respectively, one gets a decomposition $U = \begin{Vmatrix} V_1 & 0 \\ 0 & V_2 \end{Vmatrix}$ in which $V_1(t) \to 0$ for $t \to \infty$ and $V_2(t)$ is bounded not only for $t \to +\infty$, but also for $t \to -\infty$. Using the corresponding decomposition of the unit matrix

$$E = \begin{Vmatrix} E_1 & 0 \\ 0 & 0 \end{Vmatrix} + \begin{Vmatrix} 0 & 0 \\ 0 & E_2 \end{Vmatrix} = I_1 + I_2$$

into two complementary idempotents I_1, I_2, one can write

$$U = UI_1 + UI_2 = U_1 + U_2.$$

The idempotents I_1 and I_2 commute with A (and hence with U). In this form the description is independent of the coordinate system.

$$U_1(t) \to 0 \text{ for } t \to \infty, \qquad \| U_2(t) \| \leq c_2 \text{ for } t \leq 0.$$

Every vector $\mathfrak{x}$ may be split according to $I_1\mathfrak{x} + I_2\mathfrak{x} = \mathfrak{x}_1 + \mathfrak{x}_2$. For a solution $\mathfrak{z}$ of (4) the first part $\mathfrak{z}_1$ is damped, the second $\mathfrak{z}_2$ a pure sinusoidal oscillation.

Let us now decompose U in (5) into $U_1 + U_2$ and in the second part replace the lower limit of integration 0 by ∞:

$$(7) \qquad \mathfrak{x}(t) - \int_0^t U_1(t-\tau)\mathfrak{v}(\tau)\,d\tau + \int_t^\infty U_2(t-\tau)\mathfrak{v}(\tau)\,d\tau = \mathfrak{z}(t).$$

Since $\mathfrak{x}(t)$ is bounded for $t \geq 0$, (6), and

$$\| \mathfrak{v}(\tau) \| = \| \mathfrak{f}(\tau, \mathfrak{x}(\tau)) \| \leq \| \mathfrak{x}(\tau) \|\, g(\tau) \leq a^* g(\tau) \text{ for } \tau \geq 0,$$
$$\| U_2(t-\tau)\mathfrak{v}(\tau) \| \leq c_2 a^* g(\tau) \text{ for } \tau \geq t \geq 0,$$

the integral extending to infinity converges by virtue of hypothesis (III). Observe that

$$U_2(t-\tau) = U(t-\tau)I_2 = U(t)U(-\tau)I_2 = U(t)U_2(-\tau).$$

The fact that $\mathfrak{z}(t)$, (7), satisfies (4) provided $\mathfrak{x}$ is a solution of (1) is not affected by the shift of the limit of integration in the U_2-part, as one sees either by direct verification or (as Levinson does) by the simple transformation

$$U(t)\int_0^t U_2(-\tau)\mathfrak{v}(\tau)\,d\tau = U(t)\mathfrak{a}_2 - U(t)\int_t^\infty U_2(-\tau)\mathfrak{v}(\tau)\,d\tau$$

where $\mathfrak{a}_2 = \int_0^\infty U_2(-\tau)\mathfrak{v}(\tau)\,d\tau$. Therefore formula (7) associates with every solution $\mathfrak{x}$ of (1) a unique solution $\mathfrak{z}$ of (4), $\mathfrak{x} \to \mathfrak{z}$. Multiplication by I_1, I_2 splits (7) into the two equations

$$\mathfrak{x}_1(t) - \int_0^t U_1(t-\tau)\mathfrak{v}(\tau)\,d\tau = \mathfrak{z}_1(t),$$
$$\mathfrak{x}_2(t) + \int_t^\infty U_2(t-\tau)\mathfrak{v}(\tau)\,d\tau = \mathfrak{z}_2(t),$$

and thus one arrives at

LEVINSON'S THEOREM. *For every solution* $\mathfrak{x}$ *of* (1) *the part* $\mathfrak{x}_1(t)$ *tends to* 0 *with* $t \to \infty$, *and there exists a solution* $\mathfrak{z}$ *of* (4) *such that*

$$(8) \quad \mathfrak{x}_1(t) - \mathfrak{z}_1(t) = 0 \text{ for } t = 0, \qquad \mathfrak{x}_2(t) - \mathfrak{z}_2(t) \to 0 \text{ for } t \to \infty.$$

Only the statement of the first sentence remains to be proved:

$$\int_0^t U_1(t-\tau)\mathfrak{v}(\tau)\,d\tau \to 0 \text{ for } t \to \infty.$$

Split $\int_0^t$ into $\int_0^{t/2} + \int_{t/2}^t$ and, ϵ being a given positive number, assume that

$$\| U_1(t) \| \leq c_1 \text{ for } t \geq 0, \qquad \int_0^\infty g(\tau)\,d\tau = q;$$

$$\| U_1(t) \| \leq \epsilon \text{ for } t \geq t_\epsilon\ (\geq 0), \qquad \int_{t_\epsilon}^\infty g(\tau)\,d\tau \leq \epsilon.$$

Then as soon as $t \geq 2t_\epsilon$, the absolute value of the first part is less than or equal to $\epsilon \int_0^{t/2} \| \mathfrak{v}(\tau) \|\, d\tau \leq a^* q \epsilon$, that of the second part less than or equal to $c_1 \int_{t/2}^\infty \| \mathfrak{v}(\tau) \|\, d\tau \leq a^* c_1 \epsilon$.

In studying the inverse process, the transition $\mathfrak{z} \to \mathfrak{x}$, we replace (I) by the stronger Lipschitz condition

$$(\mathrm{I}^*) \qquad \mathfrak{f}(t, 0) = 0, \qquad \| \mathfrak{f}(t, \mathfrak{x}) - \mathfrak{f}(t, \mathfrak{x}^*) \| \leq \| \mathfrak{x} - \mathfrak{x}^* \| \cdot g(t),$$

which again is fulfilled in the linear case $\mathfrak{f}(t, \mathfrak{x}) = G(t)\mathfrak{x}$. For a given $\mathfrak{z}$ the integral equation (5) for $\mathfrak{x}$ may then be solved by successive approximations just as in the classical case $A = 0$. Adding Levinson's hypotheses (II) and (III) we turn at once to the more complicated integral equation (7) and show that it has a unique solution $\mathfrak{x}(t)$ provided

$$\theta = (c_1 + c_2) \int_0^\infty g(\tau)\,d\tau < 1.$$

Proof. Define the successive approximations $\mathfrak{x}_n$ by $\mathfrak{x}_0 = \mathfrak{z}$,

$$(9) \quad \mathfrak{x}_{n+1}(t) = \mathfrak{z}(t) + \int_0^t U_1(t-\tau)\mathfrak{v}_n(\tau)\,d\tau - \int_t^\infty U_2(t-\tau)\mathfrak{v}_n(\tau)\,d\tau$$

$(n = 0, 1, 2, \cdots)$ where $\mathfrak{v}_n(\tau) = \mathfrak{f}(\tau, \mathfrak{x}_n(\tau))$. Since (I^*) implies

$$\| \mathfrak{v}_n(t) - \mathfrak{v}_{n-1}(t) \| \leq \| \mathfrak{x}_n(t) - \mathfrak{x}_{n-1}(t) \| \cdot g(t)$$

(even for $n = 0$ if one sets $\mathfrak{x}_{-1}(t) = 0$), the inequality $\| \mathfrak{x}_n(t) - \mathfrak{x}_{n-1}(t) \| \leq a_n$ leads, by means of (9), to

$$\| \mathfrak{x}_{n+1}(t) - \mathfrak{x}_n(t) \| \leq \theta a_n.$$

Hence if $\| \mathfrak{z}(t) \| \leq a$ for $t \geq 0$, then

$$\| \mathfrak{x}_n(t) - \mathfrak{x}_{n-1}(t) \| \leq a\theta^n,$$

and the series $\mathfrak{x}_0 + (\mathfrak{x}_1 - \mathfrak{x}_0) + (\mathfrak{x}_2 - \mathfrak{x}_1) + \cdots$ converges at least as well as the geometric series with the quotient θ. Uniqueness is established by an argument of the same type. I like to arrange it as follows. Suppose a function $\mathfrak{x}(t)$ satisfying the equation (7) is given. Then one finds for the difference $\Delta_n \mathfrak{x} = \mathfrak{x} - \mathfrak{x}_{n-1}$:

$$\Delta_{n+1}\mathfrak{x}(t) = \int_0^t U_1(t-\tau) \cdot \Delta_n \mathfrak{v}(\tau) d\tau - \int_t^\infty U_2(t-\tau) \cdot \Delta_n \mathfrak{v}(\tau) d\tau$$

where $\Delta_n \mathfrak{v} = \mathfrak{v} - \mathfrak{v}_{n-1}$. If $\| \mathfrak{x}(t) \| \leq a^*$ the last equation yields by induction

$$\| \Delta_n \mathfrak{x}(t) \| \leq a^* \theta^n \qquad (n = 0, 1, 2, \cdots)$$

which proves the uniform convergence of $\mathfrak{x}_n(t)$ toward the given $\mathfrak{x}(t)$. We summarize:

THEOREM. *Under the hypotheses* (I*), (II), (III) *we effect the splitting* $U = U_1 + U_2$ *described above. Let*

$$\| U_1(t) \| \leq c_1 \text{ for } t \geq 0, \qquad \| U_2(t) \| \leq c_2 \text{ for } t \leq 0.$$

If t_0 *be so near to infinity that*

$$\theta = (c_1 + c_2) \int_{t_0}^\infty g(\tau) d\tau < 1$$

then we have established a one-to-one correspondence between the solutions $\mathfrak{x}$ *of* (1) *and* $\mathfrak{z}$ *of* (4) *such that*

$$\mathfrak{x}_1(t) - \mathfrak{z}_1(t) = 0 \text{ for } t = t_0, \qquad \mathfrak{x}_2(t) - \mathfrak{z}_2(t) \to 0 \text{ for } t \to \infty.$$

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ON THE CONVERGENCE OF SUCCESSIVE APPROXIMATIONS.*

By AUREL WINTNER.

Osgood's criterion for the uniqueness of the initial problem of ordinary differential equations ([3], pp. 344-345) has a dual ([5], p. 283) in which the *local* problem of *uniqueness* becomes replaced by a *non-local* problem of *existence*. In the sequel, another dual of Osgood's theorem will be exhibited, one which replaces the local problem of uniqueness of the solutions by the local problem of the convergence of the process of successive approximations.

Let $f = f(x;t)$, where f, x denote the vectors $(f_1, \cdots, f_m)$, $(x_1, \cdots, x_m)$ with real components f_i, x_i, be a continuous function of the position $(x;t)$ on an $(m+1)$-dimensional region

$$0 \leqq t \leqq a, \qquad |x| \leqq b \tag{1}$$

(the sign of the absolute value refers to Euclidean length). Then the system of m differential equations represented by $x' = f(x;t)$ has, on a sufficiently short, closed t-interval ending at $t = 0$, at least one solution $x = x(t)$ satisfying the initial condition $x(0) = 0$. On the other hand, the mere continuity of $f(x;t)$ does not ensure that the sequence of successive approximations to

$$x' = f(x;t), \qquad x(0) = 0, \tag{2}$$

that is, the sequence of the functions

$$x_{k+1}(t) = \int_0^t f(x_k(s);s)\,ds, \text{ where } x_0(t) = 0, \tag{3}$$

will converge on some (sufficiently short) interval

$$0 \leqq t \leqq c \qquad (c > 0). \tag{4}$$

Along the lines of a counter-example of Picard [4] (which concerns a boundary value problem), this has first been shown by Müller; cf. [2], pp. 35-36. And the only known estimate which does not contain t explicitly and assures the convergence of the sequence (3) on some interval (4) seems to be Lipschitz's

* Received December 17, 1945.

classical condition, requiring the existence of a constant C which satisfies the inequality

$$|f(x^*;t)-f(x^{**};t)| \leqq C\,|x^*-x^{**}| \tag{5}$$

on the region (1).

However, it turns out that, by arguments which, in every respect, are more sophisticated than the "linear" inequalities of Cauchy-Lipschitz-Picard-Lindelöf [1], the condition (5) can be improved so as to become something like a *best* restriction not containing t:

For small, positive values of r, let $\phi(r)$ be a non-negative, monotone, continuous function satisfying

$$\int_{+0} (dr)/\phi(r) = \infty \tag{6}$$

(which implies that

$$\phi(0) = 0, \tag{6 bis}$$

if $\phi(0)$ is defined to be $\phi(+0)$). Let $f(x;t)$ be a function which is given and continuous on a region (1) *and is subject there to the inequality*

$$|f(x^*;t)-f(x^{**};t)| \leqq \phi(|x^*-x^{**}|). \tag{7}$$

Then there exists an interval (4) *on which the successive approximations* (3) *to a solution $x=x(t)$ of* (2) *(exist and) are convergent (and converge, uniformly, to a solution).*

Neither of the conditions of smoothness imposed on $\phi(r)$ before (6) will be fully used and, as will be seen at the end of the paper, the first of these conditions is made superfluous by the second: *$\phi(r)$ need not be restricted to be monotone (if it is, e.g., continuous).* The parenthetical amplifications which follow (7) are trivialities, of a general nature, which have nothing to do with the assumption (7). The function f and x can be vectors.

Since (6) is satisfied when

$$\phi(r)/r = \log 1/r,\ (\log 1/r)(\log\log 1/r),\cdots, \tag{8}$$

it follows from (7) that the constant C occurring in (5) can be relaxed to any of the functions contained in the set (8). But it is worth noting that ϕ need not be an L-function. And it is just this circumstance that suggests (6) to be the ultimate condition of its kind.

Osgood's theorem, referred to above, states that, on a sufficiently short interval (4), the assumptions (6), (7) prevent the existence of more than

one path $x = x(t)$ satisfying (2) (even if ϕ is not monotone). And this time it is trivial (cf. [5], p. 284) that (6) is the best condition of its kind. Since the crucial step in the proof of the italicized theorem will involve (partly in a recondite connection, namely, via (17) below) applications of Osgood's criterion for uniqueness, one might even expect that, whether (6) and (7) be satisfied or not, *the mere uniqueness of the solution is sufficient for the convergence* of the successive approximations. However, *this was refuted* by Müller's example, quoted above, which he proved to belong to a point $(x; t) = (0; 0)$ of uniqueness. That *also the converse is false,* is shown by the familiar example of the envelope $x = 0$ of the solutions $x = x(t)$ of $x'^3 = x$.

The proof of the theorem proceeds as follows:

Supose first only that $f(x; t)$ is continuous on a region (1). Then (3) defines a sequence

$$x_1(t), x_2(t), x_3(t), \cdots \tag{9}$$

which need not converge on any interval (4). However, this sequence always is defined on an interval (4), is subject there to the inequalities $|x_k(t)| \leqq b$, and every $x_k(t)$ has there a continuous derivative satisfying $|x_k'(t)| \leqq M$, where M is a constant. Accordingly, (9) is a uniformly bounded sequence of equicontinuous functions on an interval (4). Consequently, (9) contains a subsequence, say

$$x_h(t), x_i(t), x_j(t), \cdots \qquad (h < i < j < \cdots), \tag{10}$$

which is uniformly convergent on the interval (4). For the sake of brevity, let any such subsequence of (9) be called a selected sequence. In view of the compactness of uniformly bounded, equicontinuous functions, an application of the principle of the excluded middle proves that (9) is uniformly convergent if (and, of course, only if)

$$y(t) \neq z(t) \tag{11}$$

does not occur for any t, where $y(t)$ and $z(t)$ denote the limit functions of two *unspecified* selected sequences.

If (10) is a selected sequence, then

$$x_{h+1}(t), x_{i+1}(t), x_{j+1}(t), \cdots \tag{12}$$

must be a selected sequence. In fact, since the functions (9) are uniformly bounded on the interval (8), it is clear from (3) that, if the functions (10) tend to a limit function, say $y(t)$, then the functions (12) must tend to the function

$$y^*(t) = \int_0^t f(y(s)\,;s)\,ds. \tag{13}$$

If the function represented by this integral is identical with the function $y(t)$ occurring beneath the integral sign, then it is, of course, a solution $x = x(t)$ of (2) (though not necessarily the only solution). But it can happen that this is not the case. In fact, this possibility is realized by the Picard-Müller constructions, referred to above.

It will be shown that this possibility cannot arise in the present case. In fact, it will be proved that the assumptions of the theorem imply the existence of an interval (4) on which, without any selection, the difference between two consecutive terms of the sequence (9) tends to 0; that is, on which interval the function

$$\mu(t) = \limsup |\, w_k(t)\,|, \qquad (k \to \infty), \tag{14}$$

where

$$w_k(t) = x_{k+1}(t) - x_k(t), \tag{15}$$

vanishes identically. And this will prove the theorem, since the situation is as follows:

If it is known that (15) tends to 0, then, since $y(t)$ and $y^*(t)$ in (13) denote the limit functions of two *consecutive* selected sequences, (10) and (12), these two limit functions must be identical on the interval (4). In other words, (13) becomes

$$y(t) = \int_0^t f(y(s)\,;s)\,ds,$$

which means that $x = y(t)$ is a solution of (2). But (2) cannot have more than one solution, since (6) is precisely Osgood's criterion for uniqueness. Consequently, the limit functions of all selected sequences (10) are the same. In view of the observation made in connection with (11), this implies the convergence of the sequence (9) on the interval (4).

In order to prove the identical vanishing of the function (14), put

$$\Delta g(t) = g(t + \Delta t) - g(t)$$

where $\Delta t > 0$ and $g = \mu, w_k, \cdots$. Then it is clear from (14) that

$$|\,\Delta\mu(t)\,| \leqq \limsup |\,\Delta w_k(t)\,| \qquad (k \to \infty).$$

But (3) and (15) imply that

$$|\Delta w_{k+1}(t)| \leqq \int_t^{t+\Delta t} |f(x_{k+1}(s);s) - f(x_k(s);s)|\, ds.$$

Furthermore,

$$|f(x_{k+1}(s);s) - f(x_k(s);s)| \leqq \phi(|w_k(s)|),$$

by (7) and (15). And the last three formula lines show that

$$|\Delta\mu(t)| \leqq \limsup \int_t^{t+\Delta t} \phi(|w_k(s)|)\, ds \qquad (k \to \infty).$$

If $p_1, p_2, \cdots$ are non-negative, measurable functions, then

$$\int_t^{t+\Delta t} \liminf p_k(s)\, ds \leqq \liminf \int_t^{t+\Delta t} p_k(s)\, ds \qquad (k \to \infty)$$

(Fatou). And, if C is a sufficiently large constant, this inequality is applicable to the functions

$$p_1 = C - \phi(|w_1|), \qquad p_2 = C - \phi(|w_2|), \cdots.$$

In fact, the functions (9) are uniformly bounded; hence, the same is true of the functions (15), and so the continuity of the function $\phi(r)$ assures the existence of a constant C satisfying $\phi(|w_k(t)|) < C$ on (4).

Clearly, the last three formula lines imply that

$$|\Delta\mu(t)| \leqq \int_t^{t+\Delta t} \limsup \phi(|w_k(s)|)\, ds \qquad (k \to \infty).$$

On the other hand, if t and an $\epsilon > 0$ are fixed, the definition (14) shows that there exists an N having the property that $|w_k| < \mu + \epsilon$ if $k > N$. Since $\phi(r)$ has been assumed to be a non-decreasing function, it follows that $\phi(|w_k|) \leqq \phi(\mu + \epsilon)$ if $k > N$ and so, since $\phi(r)$ is continuous,

$$\limsup \phi(|w_k|) \leqq \phi(\mu) \qquad (k \to \infty).$$

Consequently, by the preceding formula line,

$$|\Delta\mu(t)| \leqq \int_t^{t+\Delta t} \phi(\mu(s))\, ds. \tag{16}$$

The inequality (16) for the increments of the non-negative, bounded

function (14) implies that $\mu(t) \to 0$ as $t \to 0$ and that, in the sense of a parametrised Lebesgue-Stieltjes integration,

$$| d\mu | / \phi(\mu) \leqq dt, \tag{17}$$

since $\phi(t) \geqq 0$. If this differential inequality is integrated between $t = \epsilon\ (> 0)$ and a fixed $t = t_0\ (> \epsilon)$, it is seen that, if the function $\mu(t)$ did not vanish identically on an interval (4), the assumption (6) would lead to the following contradiction:

$$+ \infty \leqq t_0 - \lim_{\epsilon \to 0} \epsilon. \tag{18}$$

However, this conclusion is legitimate only if the mapping of the measures which is defined by the transformation $t \to \mu$ is sufficiently smooth; for instance, if $\mu(t)$ is continuous. In fact, if $\mu(t)$ is continuous, it must attain every μ-value contained between $\mu(+ 0) = 0$ and $\mu(t_0)$, and so the argument leading to the contradiction (18) becomes justified. But it turns out that $\mu(t)$ *is* continuous. In order to see this, it is sufficient to observe that, according to (3), (15) and (14), the function $\mu(t)$ is the upper limit of a sequence of uniformly bounded, equicontinuous functions on the interval (4). In fact, as verified by Montel in § 20-§ 21 of his Thèse (1907), the upper limit of any such sequence is always continuous.

This completes the proof of the theorem.

That the restriction of monotony, imposed on $\phi(r)$ before (6), is superfluous, follows if the consideration of the functions (15) themselves is replaced by that of their "best monotone majorant sequence," that is, of the functions

$$\operatorname*{Max}_{0 \leqq s \leqq t} | x_{k+1}(s) - x_k(s) |.$$

Corresponding to the circumstance that the above proof depends, via the criterion (11), on a "suppose the contrary" argument, there results no explicit estimate of the deviation of the solution $x(t)$ from the approximations (3). I do not know whether this is due to the *unrestricted* character of a (monotone) function ϕ allowed in (6), (7) or there can exist *general* theorems on (monotone) transformations, providing, as in the particular case (5), a direct approach.

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A REMARK ON THE MAPPING OF MULTIPLY-CONNECTED DOMAINS.*

By STEFAN BERGMAN.

1. Introduction. The fundamental theorem of the theory of conformal mapping states that the universal covering surface of a multiply-connected domain can be mapped by a conformal and one-to-one transformation on the unit circle.

Using this result it is possible to prove that every multiply connected domain satisfying certain conditions can be mapped into characteristic domains of various types.

A characteristic domain is a domain which possesses certain geometric characterising properties,[1] e. g. a ring domain with a number of slits along arcs of concentric circles. In the following, we shall refer to *this* domain as a characteristic domain $\mathfrak{C}$.

To every multiply connected domain there exist a finite number of characteristic domains $\mathfrak{C}$. Thus knowing the mapping functions of two multiply connected domains, B_k, $k = 1, 2$, into the corresponding characteristic domains, we can decide whether the domains B_k can be mapped into one another by a conformal and one-to-one transformation.

In the present note, using certain orthogonal functions, a formula for the function $w(z)$, which maps a multiply connected domain into the characteristic domain $\mathfrak{C}$ will be given.

The interest of the expression derived below lies not merely in that it can be used as a tool for the actual computing of the required mapping function, and thus make it possible actually to determine when two multiply connected domains can be mapped into each other, but that these results admit the following applications:

Using the theory of orthogonal functions,[2] introducing the kernel func-

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[1] Various types of characteristic domains were introduced by Courant [3], by Hilbert, by Koebe [5], and by Schottky [8]. Numbers in brackets refer to the bibliography.

[2] Using orthogonal functions of a complex variable the theory of multiply connected domains has been treated in a manner somewhat different from the classical approach. Instead of Poincaré's metric being introduced on the universal covering surface, a new

tion and with its aid an invariant metric, it is possible to develop a method of attack which, without using Riemann's theorem, leads to various results in the case of simply connected domains, (e. g. in the theory of distortion). See [2], [12]. Certain of these results can now be extended to the case of multiply connected domains.

One of the advantages of this approach is that it can be generalized to the case of mappings by n analytic functions of n complex variables. We do not discuss these generalizations in the present note.

2. A representation of the mapping functions of the universal covering surface into the unit circle. Let B be a bounded $(p+1)$-connected domain whose boundary, b, consists of $p+1$ regular closed curves, b_ν $(\nu = 0, 1, \cdots, p;\ p \geqq 1)$; we may suppose b_0 to be the exterior curve. It will now be shown how to determine the function which maps the universal covering surface G of the domain B into the unit circle. Let α_ν $(\nu = 1, 2, \cdots, p)$ be exterior points of the domain such that α_ν lies in the "hole" bounded by b_ν. Introduce the auxiliary domain P which consists of the entire plane, out of which have been cut the points $\alpha_1, \cdots, \alpha_p$ and the point at infinity. Let M be the universal covering surface of P; then each of the points α_ν and ∞ becomes a branch point of infinite order of M. See e. g., [7]. Clearly the universal covering surface G is a part of M.

It is well-known that there exists a function $z(\lambda)$—modular function—which maps the upper half of the λ-plane on the universal covering surface M. The inverse of this transformation, $\lambda(z)$, maps any conveniently cut sheet of M on a fundamental domain F in the λ-plane. The fundamental domain F is a simply-connected domain bounded by $2p$ semicircles in the upper half-plane which are tangent to each other and orthogonal to the real axis of the λ-plane. Thus, to each sheet of M there will correspond a fundamental domain F in the λ-plane, and it is evident that the same transformation will map the sheets of G into simply-connected portions R of the corresponding fundamental domains F. The sheets of M, and therefore those of G, may be enumerated.

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metric (which in general has a non-constant curvature) was defined for multiply connected domains, [1], [2], [9]. Further another kind of characteristic domains (so called representative and minimal domains) has been considered.

Zarankiewicz has studied the above metric in a detailed manner for the case of doubly connected domains [9], [10], and Kufareff has determined the geometric shape of minimal domains [6] in this case. Greenstone [12] has established certain interesting properties of the above metric in the general case of multiply connected domains. The image in the domain B of the Cartesian coordinate frame of the representative domain of B is termed the "representative coordinate frame" of B. In [11] and papers cited there Fuchs has studied properties of representative coordinate systems.

Let G_k be the sheet of G which lies on the sheet M_k of M. Suppose that the sheets G_k have been so joined and enumerated that $\sum_{k=1}^{n} G_k$ is simply connected for every n. That this may always be done is obvious.[3]

The function $\lambda(z)$ maps G_1 into the domain R_1 which lies in the upper half of the λ-plane, as already described. Let λ_0 be an interior point of R_1; and let $w_1(\lambda)$ be the function which maps R_1 into the unit circle, taking λ_0 into the origin. $\sum_{k=1}^{n} G_k$ is a simply-connected domain which is mapped by $\lambda(z)$ into $\sum_{k=1}^{n} R_k$, which is also a simply-connected domain containing λ_0 in its interior. Therefore there exists a function $w_n(\lambda)$ which maps $\sum_{k=1}^{n} R_k$ into the unit circle, taking λ_0 into the origin. $G = \sum_{k=1}^{\infty} G_k$ is the kernel domain (in the sense of Carathéodory) of the sequence of domains $\sum_{k=1}^{n} G_k$, and therefore, by the theorem of Carathéodory [3], $\lim_{n\to\infty} w_n[\lambda(z)]$ exists and maps G into the unit circle. Thus,

$$w(z) = \lim_{\nu\to\infty} w_\nu[\lambda(z)] \tag{2.1}$$

represents the required mapping function which maps G into the unit circle.

The function (2.1) can immediately be expressed in terms of orthogonal functions.

Indeed, denote by

$$\{\Psi_k^{(\nu)}\}, \qquad (k = 1, 2, 3, 4, \cdots) \tag{2.2}$$

the set of orthogonal functions which we obtain by orthogonalizing the powers λ^k $(k = 0, 1, 2, \cdots)$, with respect to $\sum_{n=1}^{\nu} G_n$. Then the function $w_\nu(\lambda)$ which maps $\sum_{n=1}^{\nu} G_n$ into the unit circle can be written in the form

$$w_\nu(\lambda) = \frac{\pi^{\frac{1}{2}} \int_0^\lambda \sum_{k=1}^{\infty} \Psi_k^{(\nu)}(\lambda)\,\overline{\Psi_k^{(\nu)}(\lambda_0)}\,d\lambda}{\left[\sum_{k=1}^{\infty} |\Psi_k^{(\nu)}(\lambda_0)|^2\right]^{\frac{1}{2}}}. \tag{2.3}$$

The formulae (2.1), (2.3) yield the mapping of the universal covering surface into the unit circle in the w-plane. The functions which map B into characteristic domains can be expressed in a well-known manner, in the form of an infinite series in terms of $w(z)$. See e. g. [5].

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[3] If any two simply connected domains, say G_1 and G_2, have a simple open curve as their only common boundary, then the combined domain $G_1 + G_2$ is also simply connected. This is exactly the case we have here, because G_1 and G_2 are joined along one

On the other hand the inverse of the modular function appears in (2.1), (2.3); further (2.1) is expressed not directly in terms of orthogonal functions, but as a limit of a series development of this kind. This fact often causes certain difficulties, e. g. in actual computation of the mapping function, in the distortion theory, etc. In the next section we shall derive an expression for the mapping function $w(z)$ of B into the characteristic domain $\mathfrak{C}$. The formula obtained avoids the above disadvantages; we express $w(z)$ in terms of certain functions which we obtain by orthogonalizing the powers z^n $(n = 0, 1, 2, \cdots)$, $(z - \alpha_\nu)^{-h}$ $(\nu = 1, 2, \cdots, p,\ h = 1, 2, \cdots)$. Here α_ν are the points introduced above. On the other hand in the formula certain unknown constants appear. These constants can be determined by certain additional considerations, which are however not discussed in detail in the present paper.

3. The determination of an analytic function from boundary values of its real part. Let B be the domain introduced in **2**, whose boundary b consists of $p+1$ regular closed curves b_ν $(\nu = 0, 1, 2, \cdots, p,\ p \geqq 1)$.

In the following we shall denote by $\{\phi'_k(z)\}$ $(k = 1, 2, 3, \cdots)$, a complete system of orthogonal functions, i. e. a system for which

$$(3.1) \qquad \iint_B \phi'_k(z)\overline{\phi'_s(z)}\,d\omega = 1 \text{ for } k = s,\ d\omega = dxdy,$$
$$= 0 \text{ for } k \neq s,$$

and such that for every analytic function $g(z)$, $\iint_B |\, g(z)|^2 d\omega < \infty$,

$$(3.2) \qquad \iint_B |\, g(z)|^2\, d\omega = \sum_{k=1}^{\infty} \Big| \iint_B g(z)\overline{\phi'_k(z)}\,d\omega \Big|^2.$$

Let a be an interior point of B. We shall consider in the following the functions

$$(3.3) \qquad \phi_k(z) = \psi_k(x, y) + i\theta_k(x, y), \qquad \phi_k(a) = 0.$$

Remark. Since B is a multiply connected domain the functions $\phi_k(z)$ are multivalued. See examples (3.10) and (3.11). In order to avoid any ambiguity, we introduce the simply connected domain $\tilde{B}$ obtained from B by p conveniently chosen cuts. The functions $\phi_k(z)$ are single valued in $\tilde{B}$.

THEOREM 3.1. *Let $f(z) = f_1(x, y) + if_2(x, y)$ be a function which is regular in B, whose derivative $f'(z)$ as well as the real part f_1 are single valued*

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of the simple curves along which B has been cut. By induction it follows immediately that this holds for any number of simply connected domains.

in B. $f(z)$ is assumed to map B into a domain with a finite area, i. e. it is assumed that

$$\iint_B |f'(z)|^2 \, d\omega < \infty. \tag{3.4}$$

Then $f(z)$ can be represented in $\tilde{B}$ in the form

$$f(z) = i\sum_{k=1}^{\infty} \phi_k(z) \int_b f_1(\xi,\eta)\, d\overline{\phi_k(\xi+i\eta)} + f(a), \qquad z \in \tilde{B}. \tag{3.5}$$

Proof. By the theorem on the development in the series of orthogonal functions, see [2] p. 47, we can write

$$f'(z) = \sum_{k=1}^{\infty} \phi'_k(z) \iint_B f'(Z)\overline{\phi'_k(Z)}\, d\Omega, \quad d\Omega = dXdY, \quad Z = X + iY. \tag{3.6}$$

Now

$$\begin{aligned}
\iint_B f'(Z)\overline{\phi'_k(Z)}\, d\Omega &= \iint_{\tilde{B}} f'(Z)\overline{\phi'_k(Z)}\, d\Omega \\
&= \iint_{\tilde{B}} [f_{1,X}\psi_{k,X} + f_{2,X}\theta_{k,X}]\, d\Omega + i\iint_{\tilde{B}} [f_{2,X}\psi_{k,X} - f_{1,X}\theta_{k,X}]\, d\Omega \\
&= \iint_{\tilde{B}} [f_{1,X}\psi_{k,X} + f_{1,Y}\psi_{k,Y}]\, d\Omega - i\iint_{\tilde{B}} [f_{1,X}\theta_{k,X} + f_{1,Y}\theta_{k,Y}]\, d\Omega \\
&= \int_b f_1 \left[-\frac{\partial\psi_k}{\partial n}\, ds + i\frac{\partial\theta_k}{\partial n}\, ds\right] \\
&\quad + \sum_{\nu=1}^{2p} \int_{e_\nu} f_1 \left[-\frac{\partial\psi_k}{\partial n}\, ds + i\,\frac{\partial\theta_k}{\partial n}\, ds\right],
\end{aligned} \tag{3.7}$$

$f_{1,X} \equiv (\partial f_1/\partial X)$, $\psi_{k,X} \equiv (\partial\psi_k/\partial X)$ etc.

where $e_{2\nu}$, $e_{2\nu-1}$ $(\nu = 1, 2, \cdots, p)$, are the edges of crosscuts which dissect B into a simply connected domain $\tilde{B}$. dn and ds denote an element of the interior normal and the line element, respectively. Since f_1, $\partial\psi_k/\partial n$ and $\partial\theta_k/\partial n$ are single valued in B, the integrals over e_ν vanish and we find that (3.7) equals

$$\int_b f_1[d\theta_k + i d\psi_k] = i\int_b f_1(\xi,\eta)\, d\overline{\phi_k(\zeta)}. \tag{3.8}$$

A system of orthogonal functions satisfying the above conditions can be obtained by orthogonalizing the system

$$1,\ z^k,\ (z-\alpha_\mu)^{-k}, \qquad (\mu = 1, \cdots, p,\ k = 1, 2, \cdots) \tag{3.6}$$

where α_μ is a point which lies in the "hole" bounded by b_μ. See [2], p. 57. In this case the functions ϕ_k are of the following form:

$$(3.10)\qquad \begin{aligned}
&\phi_1 = \beta_{1,0} + \beta_{1,1}z,\\
&\phi_2 = \beta_{2,0} + \beta_{2,1}z + \beta_{2,2}z^2,\\
&\phi_{3+\mu} = \beta_{3+\mu,0} + \beta_{3+\mu,1}z + \beta_{3+\mu,2}z^2 + \sum_{h=0}^{\mu}\beta_{3+\mu,3+h}\log(z-\alpha_h), \qquad (\mu = 0,1,\cdots,p),\\
&\phi_{4+p} = \beta_{4+p,0} + \beta_{4+p,1}z + \beta_{4+p,2}z^2 + \sum_{h=0}^{p}\beta_{4+p,3+h}\log(z-\alpha_h) + \beta_{4+p,4+p}z^2,\\
&\phi_{5+p+s} = \beta_{5+p+s,0} + \beta_{5+p+s,1}z + \beta_{5+p+s,2}z^2\\
&\qquad + \sum_{h=0}^{p}\beta_{5+p+s,3+h}\log(z-\alpha_h) + \beta_{5+p+s,4+p}z^3\\
&\qquad + \sum_{g=0}^{s}\beta_{5+p+s,5+p+g}(z-\alpha_g)^{-1}, \qquad (s = 0,1,\cdots,p),
\end{aligned}$$

Example. If $B = \mathrm{E}[r < |z| < 1]$ then $p = 1$. We choose $\alpha_1 = 0$.

$$(3.11)\qquad \begin{aligned}
&\phi_1 = (z-a)/[\pi(1-r^2)]^{\frac{1}{2}}, \qquad \phi_3 = (\log z - \log a)(-2\pi\log r)^{-\frac{1}{2}},\\
&\phi_{2n-2} = (z^n - a^n)[\pi n(1-r^{2n})]^{-\frac{1}{2}}, \qquad (n = 2,3,\cdots),\\
&\phi_{2n-1} = (a^{1-n} - z^{1-n})[\pi(1-n)(1-r^{2(n-1)})]^{-\frac{1}{2}}, \quad (n = 3,4,\cdots).
\end{aligned}$$

4. The representation of integrals of the first kind in terms of orthogonal functions. The domain B being given, and two boundary curves, b_1 and b_0, being prescribed there exists a function $w(z)$ which maps B into $\mathfrak{C}$ i. e. the circular ring.[4] $\mathrm{E}[\exp a_1 < |z| < 1]$ with $(p-1)$ slits along arcs $\mathrm{E}[\alpha_{\nu 1} < \arg z < \alpha_{\nu 2},\ |z| = \exp a_\nu]$ $(\nu = 2,\cdots,p)$, $a_1 < a_2 \leqq a_3 \cdots \leqq a_{p-1} < 1$. See [5]. Two prescribed components, b_0 and b_1 of the boundary b of B are transformed into the interior and exterior circles, respectively.

In this section we shall express the functions

$$(4.1)\qquad s(z) = [\log w(z)]$$

in terms of functions $\phi_k(z)$, introduced in (3.9).

The function $s(z)$ possesses the following properties:

I. (4.2) $\mathrm{Re}\,[s(z)] = a_\nu$ on b_ν, $\quad (\nu = 0,1,\cdots,p-1,\ a_0 = 0)$.

II. The function $s(z)$ maps the conveniently cut domain B into the rectangle

$$\mathrm{E}[a_1 < \xi < 0,\ 0 < \eta < 2\pi]$$

[4] By E [...] we denote a set of points whose coordinates satisfy the inequalities indicated in brackets.

with $(p-1)$ slits along the straight lines $\xi = a_\nu$ $(\nu = 1, 2, \cdots, p-1)$. (These slits are not necessarily connected.)

III. The derivative $s'(z)$ as well as $\mathrm{Re}[s(z)]$ are single valued functions in B.

THEOREM 4.1. *The function $s(z)$ can be represented in B in the form*

$$s(z) = i \sum_{k=1}^{\infty} \phi_k(z) \sum_{\mu=1}^{p} a_\mu \Delta_\mu \overline{\phi_k(z)} + s(a) \tag{4.3}$$

where $\Delta_\mu \overline{\phi_k(z)}$ denotes the increase of the function $\overline{\phi_k(z)}$ as we move around b_μ $(\mu = 0, 1, 2, \cdots, p-1)$.

Proof. Since $s(z)$ satisfies all the hypotheses of Theorem 3.1, we can write it in the form (3.5). In the case under consideration we have

$$\int_b f_1 d\bar{\phi}_k = \sum_{\mu=0}^{p} a_\mu \int_{b_\mu} d\bar{\phi}_k = \sum_{\mu=1}^{p} a_\mu \Delta_\mu \bar{\phi}_k \tag{4.4}$$

from which (4.3) follows. (Note that $a_0 = 0$).

Example. Suppose $B = \mathrm{E}[r < |z| < 1]$ and $\alpha_1 = 0$. According to (3.11) all $\Delta\phi_k = 0$, except for $k = 3$, for which we have

$$\Delta\phi_3 = (-2\pi \log r)^{-\frac{1}{2}} \cdot 2\pi i,$$

and therefore

$$s(z) = i \log r \, (-2\pi \log r)^{-1} \cdot 2\pi i (\log z - \log a) + \log a = \log z.$$

Remark. Except $\log(z - \alpha_\mu)$ all expressions in (3.7) do not change if we move around b_μ. Hence

$$\Delta_\mu \phi_k = -2\pi i \beta_{k,\mu+3} \tag{4.5}$$

where $\beta_{k,\mu}$ are the quantities introduced in (3.7).

Remark. p unknown constants a_μ appear in the expression (4.3). We note that the theory of orthogonal functions provides the means for determining them. The procedure is based on the fact that since $\mathrm{Re}[s(z)] = a_\mu$ on b_μ $(\mu = 1, 2, \cdots, p)$,

$$\mathrm{Re}[\lim \int_{z_1}^{z_2} s(z)\, dz] = a_{\mu_2} - a_{\mu_1}.$$

Here z_1 and z_2 are points which lie in B and are such that $\lim z_k \epsilon b_{\mu_k}$ $(k = 1, 2)$. We omit a more detailed determination of the constants here.

5. An application to the theory of distortion. As is indicated in more detail in [2] § 9 the formulae expressing functions in terms of orthogonal functions often yield inequalities for the functions under consideration. As an example of the application of these methods, an inequality for $s'(z) = d[\log w(z)]/dz$ will be derived in this section.

THEOREM 5.1. *Let c_ν, $c_\nu \in B$, be p closed simple curves with the following properties:*

1. *The distance of every point of c_ν from the boundary b of B is greater than δ_1.*

2. *By a one-to-one and continuous transformation each c_ν can be reduced to b_ν without cutting the remaining b_μ, $\mu \neq \nu$.*

Let $w(z)$ be the function which maps B into the characteristic domain $\mathfrak{G}$ i.e. $\mathrm{E}[\exp a_1 < |z| < 1]$ with $p-1$ slits along arcs situated on circles $|z| = \exp a_\nu$ $(\nu = 2, 3, \cdots, p)$.

Further let z be a point of B, whose distance from the boundary is larger than δ_2. Then

$$\left|\frac{d \log w(z)}{dz}\right| \leqq \sum_{\nu=1}^{p} \frac{|a_\nu| \, l(c_\nu)}{\pi \delta_1 \delta_2} \tag{5.1}$$

where $l(c_\nu)$ denotes the length of c_ν.

Proof. According to (3.5) and (4.4)

$$\frac{d \log w(z)}{dz} = \frac{ds(z)}{dz} = i \sum_{k=1}^{\infty} \phi'_k(z) \sum_{\nu=0}^{p} a_\nu \int_{b_\nu} \overline{\frac{d\phi_k(\zeta)}{d\zeta}} \, d\bar{\zeta}. \tag{5.2}$$

From 2. it follows that the integration curves b_ν can be replaced by the c_ν. Thus (5.2) equals [5]

$$\begin{aligned} & i \sum_{k=1}^{\infty} \phi'_k(z) \sum_{\nu=0}^{p} a_\nu \int_{c_\nu} \overline{\frac{d\phi_k(\zeta)}{d\zeta}} \, d\bar{\zeta} \\ &= i \sum_{\nu=1}^{p} a_\nu \int_{c_\nu} \sum_{k=1}^{\infty} \phi'_k(z) \overline{\phi'_k(\zeta)} \, d\bar{\zeta} \\ &= i \sum_{\nu=1}^{p} a_\nu \int_{c_\nu} \mathrm{K}_B(z, \bar{\zeta}) \, d\bar{\zeta}. \end{aligned} \tag{5.3}$$

Here $\mathrm{K}_B(z, \bar{\zeta})$ denotes the kernel function of B. See [2], § 7. From hypothesis 1, the assumption concerning the location of z and (7.18) p. 47 of [2],

[5] By the theorem on p. 47 of [2] the series $\sum_{\nu=1}^{\infty} |\phi'_\nu(z) \phi'_\nu(\zeta)|$ converges uniformly in every closed domain which lies in B.

$$K_B(z,\bar{\zeta}) \leqq [\sum_{\nu=1}^{\infty} |\phi'_\nu(z)|^2 \cdot \sum_{\nu=1}^{\infty} |\phi'_\nu(\zeta)|^2]^{\frac{1}{2}} \leqq \pi^{-1}\delta_1^{-1}\delta_2^{-1},$$

which yields the inequality (5.1).

We note that using other inequalities for $K_B(z,\bar{\zeta})$ and for its derivatives (see [2] § 9) one can obtain other inequalities similar to (5.1).

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ON A PROBLEM OF RAMANUJAN.*

By Arnold E. Ross.

1. It was apparently known to Diophantus and first proven by Lagrange (8) that the form $x^2 + y^2 + z^2 + u^2$ represents all positive integers. Examples of other integral forms

$$\Phi = ax^2 + by^2 + cz^2 + du^2 \tag{1.1}$$

which represent all positive integers, were first obtained by Jacobi (5), Liouville (9), and Pepin (13). Ramanujan (14) proved that there are only 54 sets of positive integers a, b, c, d such that (1.1) represents all positive integers. Dickson (2) called such forms *universal*. Universal quaternary quadratic forms with cross products were studied by Dickson (2) and Morrow (11).

In the above mentioned paper Ramanujan proposed another problem, *viz.*, the problem of determining the conditions under which positive quadratic forms (1.1) represent all except a finite number of integers. Kloosterman (7), employing the methods of Hardy-Littlewood succeeded, save for a finite number of exceptions, in solving that problem.

It is natural to ask Ramanujan's question concerning general positive quaternary quadratic forms. Should Tartakowsky's theorem (19) concerning the representation of large integers by positive quadratic forms in $n \geqq 5$ variables hold also for $n = 4$, then one would expect the answer to that question to be found as an elementary corollary of this theorem and to be expressed in terms of the generic characters of quadratic forms. It is of interest to note that, although Tartakowsky's theorem does not carry over unconditionally to forms in four variables, still for forms of odd determinants and certain orders of even determinants, the answer to Ramanujan's question may be obtained as an elementary extension of the results of Kloosterman and some other elementary considerations, and that, moreover, save for a finite number (of classes) of exceptions the conditions are given in terms of the generic characters. The results here obtained suggest conditions which the generic characters of a genus of quaternary forms should fulfil in order that all forms of that genus should represent the same large integers.

The method employed may be summarized as follows: Through the use of the canonical form of Section **3**, the problem of the representation of in-

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tegers by the original form is reduced to that of the representation of integers by a certain quadratic form without the cross products (Section **4.** 1). Kloosterman's conditions (7) applied to this last form (Section **5**) yield a set of generic character conditions which assure the representation of all large integers by the original form. Upon closer examination of these conditions one notices (Sections **6** and **7**) that some of these are necessary but that the failure of the remaining merely implies that a form represents *all large* integers only if it represents *all* integers or *all even* integers. In view of the results in Section **2**, the determinants of such forms do not exceed a fixed number. Thus, outside of forms in Section **5**, there is only a finite number of classes of forms representing all large integers. A study of some of these classes (Section **8**) yields interesting examples of representation of integers by positive quaternary forms in a fixed genus.

2. An upper bound for determinants of classic universal positive quaternary quadratic forms. Among the 54 universal forms of type (1. 1), the form $x^2 + 2y^2 + 4z^2 + 14u^2$ has [1] the largest determinant 112. A simple extension of Ramanujan's argument yields [2] the more general and quite useful

THEOREM 2. *The determinant of every classic universal positive quaternary quadratic form is* $\leqq 112$.

We write

$$\Phi_1(x) = x'Ax = \sum_{i,j=1}^{4} a_{ij}x_ix_j \tag{2.01}$$

where a_{ij} are integers. If $\Phi_1(x)$ represents all positive integers, it represents 1 properly, and hence is equivalent to a form of type (2. 01) with $a_{11} = 1$ and $a_{1j} = 0$ for $j = 2, 3, 4$. Thus

$$\Phi_1(x) \sim \Phi_2(y) = y_1^2 + \phi_2(y_2, y_3, y_4) = y_1^2 + \sum_{i,j=2}^{4} b_{ij}y_iy_j.$$

In order that Φ_1 and, hence, Φ_2 should represent all positive integers, the minimum a of ϕ_2 must be ≤ 2. For otherwise Φ_2, and therefore also Φ_1, would not represent 2. Since the minimum a is represented properly by ϕ_2

$$\phi_2 \sim \phi_3 = az_2^2 + bz_3^2 + cz_4^2 + 2rz_3z_4 + 2sz_2z_4 + 2tz_2z_3, \tag{2.02}$$

where

$$0 \leq s < a \quad \text{and} \quad 0 \leq t < a, \qquad a = 1 \text{ or } 2, \tag{2.03}$$

and hence

$$\Phi_1 \sim \Phi_3 = z_1^2 + \phi_3(z_2, z_3, z_4). \tag{2.04}$$

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[1] Cf. Dickson (4), p. 115.

[2] Ross (15), Theorem 8.

2.1. We let, first, $a = 1$. Then, in view of (2.02)-(2.04),

$$\Phi_3 = z_1^2 + z_2^2 + bz_3^2 + cz_4^2 + 2rz_3z_4 = z_1^2 + z_2^2 + \psi_3(z_3, z_4). \tag{2.11}$$

In order that Φ_3 should represent 3, the minimum M of ψ_3 should be ≤ 3. Then

$$\psi_3 \sim \psi_4 = Mu_3^2 + 2Nu_3u_4 + Lu_4^2, \tag{2.12}$$

$$-(M/2) < N \leq M/2, \qquad M = 1, 2, \text{ or } 3, \tag{2.13}$$

and

$$\Phi_1 \sim \Phi_4 = u_1^2 + u_2^2 + \psi_4. \tag{2.14}$$

The form Φ_1 would represent all integers only if $M\Phi_1$ should represent all multiples of M. But in view of (2.12)-(2.13)

$$M\Phi_1 \sim M\Phi_4 = Mu_1^2 + Mu_2^2 + (Mu_3 + Nu_4)^2 + Du_4^2$$

where $D = ML - N^2$ is the determinant of Φ_1. Thus in order that $M\Phi_1$ should represent all multiples of M, D must not exceed the smallest multiple of M not represented by

$$f_M(u_1, u_2, U_3) = Mu_1^2 + Mu_2^2 + U_3^2.$$

If $M = 1$, $f_1 = u_1^2 + u_2^2 + U_3^2 \neq 7$ and hence $D \leq 7$.
If $M = 2$, $f_2 = 2u_1^2 + 2u_2^2 + U_3^2 \neq 28$ and therefore $D \leq 28$.
If $M = 3$, $f_3 = 3u_1^2 + 3u_2^2 + U_3^2 \neq 18$ and therefore $D \leq 18$.

Thus, in case $a = 1$, there is no universal form Φ_1 of determinant > 28.

2.2. Next, let $a = 2$. In order that Φ_1 should be universal $2\Phi_1$ should represent all even integers. But in view of (2.02) and (2.04),

$$2\Phi_1 \sim 2\Phi_3 = 2z_1^2 + (2z_2 + tz_3 + sz_4)^2 + \psi_3(z_3, z_4)$$

where

$$\psi_3(z_3, z_4) = (ab - t^2)z_3^2 + 2(ar - st)z_3z_4 + (ac - s^2)z_4^2.$$

Since $2z_1^2 + Z_2^2 \neq 10$, the minimum M of $\psi_3(z_3, z_4)$ is ≤ 10. Also,

$$\psi_3 \sim \psi_4 = Mu_3^2 + 2Nu_3u_4 + Lu_4^2$$
$$2\Phi_1 \sim 2\Phi_4 = 2u_1^2 + (2u_2 + t_1u_3 + s_1u_4)^2 + \psi_4(u_3, u_4)$$

and

$$2M\Phi_1 \sim 2M\Phi_4 = 2Mu_1^2 + M(2u_2 + t_1u_3 + s_1u_4)^2 + (Mu_3 + Nu_4)^2 + (ML - N^2)u_4^2.$$

In order that Φ_1 should be universal $2M\Phi_1$ should represent all multiples of $2M$. Hence $ML - N^2$ does not exceed the smallest multiple of $2M$ not represented by

$$f_M(u_1, U_2, U_3) = 2Mu_1^2 + MU_2^2 + U_3^2.$$

Since by a well known theorem on determinants the determinant D of Φ_1 is equal to $\frac{1}{2}(ML - N^2)$, we have the following results which we state in a schematic form:

M	$f_M = 2Mu_1^2 + MU_2^2 + U_3^2 \neq 2Mk$	$ML - N^2 \leq 2Mk$	$D \leq$
1	$2u_1^2 + U_2^2 + U_3^2 \neq 14$	≤ 14	≤ 7
2	$4u_1^2 + 2U_2^2 + U_3^2 \neq 56$	≤ 56	≤ 28
3	$6u_1^2 + 3U_2^2 + U_3^2 \neq 30$	≤ 30	≤ 15
4	$8u_1^2 + 4U_2^2 + U_3^2 \neq 56$	≤ 56	≤ 28
5	$10u_1^2 + 5U_2^2 + U_3^2 \neq 50$	≤ 50	≤ 25
6	$12u_1^2 + 6U_2^2 + U_3^2 \neq 120$	≤ 120	≤ 60
7	$14u_1^2 + 7U_2^2 + U_3^2 \neq 98$	≤ 98	≤ 48
8	$16u_1^2 + 8U_2^2 + U_3^2 \neq 224$	≤ 224	≤ 112
9	$18u_1^2 + 9U_2^2 + U_3^2 \neq 126$	≤ 126	≤ 63
10	$20u_1^2 + 10U_2^2 + U_3^2 \neq 200$	≤ 200	≤ 100

Thus in every case the determinant D does not exceed 112.

2.3. We have just seen that there is but a finite number of classes of classic universal positive quaternary quadratic forms. We inquire next whether the same would be true of forms which, although not universal, do nevertheless represent all even integers. We show that

THEOREM 2.3. *The determinants of classic positive quaternary quadratic forms which represent all even integers, do not exceed a fixed upper bound* B_2.

Let $\Phi_1(x)$ in (2.01) represent all even integers. Let a_{11} be the minimum of Φ_1. Then

(2.31) $$a_{11} \leq 2.$$

Next

$$a_{11}\Phi_1 = \Phi_3 = z_1^2 + \phi_3(z_2, z_3, z_4)$$

where ϕ_3 is given by (2.02). Let a be the minimum of ϕ_3. Then, it is easily seen that

(2.32) $$a \leq \beta(a_{11})$$

where $\beta(a_{11})$ may be taken as the least multiple of $2a_{11}$, which is not a square. Proceeding further we see that

$$aa_{11}\Phi_1 = au_1^2 + u_2^2 + \Psi_4(u_3, u_4)$$

where Ψ_4 is given by (2.12). Let M be the minimum of Ψ_4. Then

(2.33) $$M \leq \Delta(a_{11}, a)$$

where $\Delta(a_{11}, a)$ may be taken as the least multiple of $2a_{11}a$ which is not represented by $au_1^2 + u_2^2$. Next

$$Maa_{11}\Phi_1 = Mau_1^2 + Mu_2^2 + U_3^2 + (ML - N^2)u_4^2$$

where

$$ML - N^2 = aa_{11}^2 D, \qquad D = |a_{ij}|, \tag{2.341}$$

and

$$D \leq ML - N^2 \leq B(M, a, a_{11}) \leqq B_2 \tag{2.342}$$

where $B(M, a, a_{11})$ may be taken as a multiple of $2M \cdot a \cdot a_{11}$ which is not represented by

$$Mau_1^2 + Mu_2^2 + U_3^2. \tag{2.34}$$

We are now ready to prove that B_2 in (2.342) is an absolute constant.

One may easily verify that $\beta(1) = 2$ and $\beta(2) = 8$. Examining the form $au_1^2 + u_2^2$ we get the following values for $\Delta(a_{11}, a)$ in (2.33)':

a_{11}	1	1	2	2	2	2	2	2	2	2
a	1	2	1	2	3	4	5	6	7	8
$\Delta(a_{11}, a)$	6	20	12	40	24	48	40	48	84	160

To extract the best value of B_2 out of the above inequalities one should determine the best value of $B(M, a, a_{11})$ for each set of values M, a_1, a_{11} permitted by this table. Although this presents no difficulty, the computation is somewhat lengthy and we shall therefore merely prove the existence of such an upper bound. To do this it suffices, in view of the above discussion, to show that in every case the ternary form in (2.34) will fail to represent a multiple of $2Maa_{11}$. This last follows at once from a result due to Hasse.[3]

3. The canonical form (C_p). We shall find the following normalization useful in the subsequent discussion.

THEOREM 3. *Every properly primitive classic quaternary quadratic form with integral coefficients and invariants*[4] o_μ *is equivalent to a canonical form*

$$f = \Sigma a_{ij}x_i x_j = x'Ax \tag{3.01}$$

of determinant $|A| = |a_{ij}|$ *whose leading principal minors are*

$$a_{11} = A_1, \qquad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = o_1 A_2, \qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = o_1^2 o_2 A_3$$

[3] Hasse (6a), § 11.

[4] After Minkowski and Smith. We employ the notation of Minkowski.

where

(C_p) $\qquad A_\mu$ *or* $\frac{1}{2}A_\mu$ *is an odd prime not dividing* $|\,a_{ij}\,|\,A_kA_l$
$(\mu, k, l) = (1, 2, 3),\ (2, 1, 3),\ (3, 1, 2).$

Since our form is properly primitive we may assume at once that a_{11} is an odd prime not dividing $|\,a_{ij}\,|$. Write $o_1{}^2o_2A_{ij}$ for the algebraic complement of a_{ij} in $|\,a_{ij}\,|$. Then $F = \Sigma A_{ij}X_iX_j$ is the reciprocal [5] of f. In view of the choice of a_{11} the ternary section $F(0, X_2, X_3, X_4)$ of F is a primitive ternary form of invariants [6]

(3.03) $\qquad \Omega = o_3 \quad \text{and} \quad \Delta = o_2a_{11}.$

This ternary, however, is equivalent to a form whose third coefficient A_{44} $(= A_3)$ and the leading coefficient $(= A_2)$ of whose reciprocal are distinct odd primes not dividing $o_1o_2o_3A_1$ or doubles of such primes.[7] Replacement of the ternary section by this canonical form does not disturb [8] the choice of a_{11}.

4. The associates of a given quadratic form. In this section we assume that our quadratic form (2.01) is a canonical form (C_p). Multiplying through by $a_{11} = A_1$ we obtain

(4.01) $\qquad A_1f = X_1{}^2 + o_1\sum_{i,j=2}^{4}\alpha_{ij}{}^{(1)}x_ix_j,$

where

(4.02) $\quad o_1\alpha_{ij}{}^{(1)} = \begin{vmatrix} a_{11} & a_{1j} \\ a_{i1} & a_{ij} \end{vmatrix} \quad \alpha_{22}{}^{(1)} = A_2,\ X_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4.$

Next

(4.03) $\qquad A_2A_1f = A_2X_1{}^2 + o_1(X_2{}^2 + a_{11}o_2\sum_{i,j=3}^{4}\alpha_{ij}{}^{(2)}x_ix_j),$

where, in view of a determinant theorem of Sylvester,

(4.04) $$\begin{vmatrix} \alpha_{22}{}^{(1)} & \alpha_{2j}{}^{(1)} \\ \alpha_{i2}{}^{(1)} & \alpha_{ij}{}^{(1)} \end{vmatrix} = \frac{1}{o_1{}^2}\begin{vmatrix} o_1\alpha_{22}{}^{(1)} & o_1\alpha_{2j}{}^{(1)} \\ o_1\alpha_{i2}{}^{(1)} & o_1\alpha_{ij}{}^{(1)} \end{vmatrix}$$
$$= \frac{a_{11}}{o_1{}^2}\begin{vmatrix} a_{11} & a_{12} & a_{1j} \\ a_{21} & a_{22} & a_{2j} \\ a_{i1} & a_{i2} & a_{ij} \end{vmatrix} = \frac{a_{11}o_1{}^2o_2\alpha_{ij}{}^{(2)}}{o_1{}^2} = a_{11}o_2\alpha_{ij}{}^{(2)}$$

(4.05) $\qquad \alpha_{33}{}^{(2)} = A_3 \quad \text{and} \quad X_2 = \alpha_{22}{}^{(1)}x_2 + \alpha_{23}{}^{(1)}x_3 + \alpha_{24}{}^{(1)}x_4.$

Finally

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[5] Cf. Dickson (3).

[6] Minkowski (10), Ch. XVIII.

[7] Ross (16). The precise statement of the theorem referred to implies that if Ω, Δ are odd A_2 and A_3 may be taken as odd primes.

[8] Cf. Dickson (3).

(4.06) $$A_3A_2A_1f = A_3A_2X_1^2 + o_1A_3X_2^2 + o_1o_2A_1X_3^2 + o_1o_2o_3A_1A_2X_4^2,$$

since

(4.07) $$\begin{vmatrix} \alpha_{23}^{(2)} & \alpha_{34}^{(2)} \\ \alpha_{43}^{(2)} & \alpha_{44}^{(2)} \end{vmatrix} = \frac{1}{(o_1^2o_2)^2} \begin{vmatrix} o_1^2o_2\alpha_{33}^{(2)} & o_1^2o_2\alpha_{34}^{(2)} \\ o_1^2o_2\alpha_{43}^{(2)} & o_1^2o_2\alpha_{44}^{(2)} \end{vmatrix}$$

$$= \frac{o_1\alpha_{22}^{(1)} \mid A \mid}{o_1^4o_2^2} = \frac{A_2o_1^4o_2^2o_3}{o_1^4o_2^2} = o_3A_2.$$

Here

(4.08) $$X_3 = \alpha_{33}^{(2)}x_3 + \alpha_{34}^{(2)}x_4, \quad X_4 = x_4.$$

We now introduce the form

(4.09) $$G(X_1, X_2, X_3, X_4) = A_3A_2X_1^2 + o_1A_3X_2^2 + o_1o_2A_1X_3^2 + o_1o_2o_3A_1A_2X_4^2$$

in the independent variables X_1, X_2, X_3, X_4. We shall call G the *associate* of f.

4.1. The form G is of interest by virtue of the following:

THEOREM 4.1. *Let f be a properly primitive quaternary quadratic form. Employ the notation of Theorem 3 and assume that f is in the canonical form of type (C_p). Then if f represents an integer m its associate G in* (4.09) *represents $A_1A_2A_3m$. Conversely, if the form G represents $A_1A_2A_3m$ and* (4.14) *and* (4.16) *hold, then the original form f represents m.*

The first part of the theorem is trivial, for if x_1, x_2, x_3, x_4 are integers, then by (4.02), (4.05), and (4.08), so also are X_1, X_2, X_3, X_4 and $A_1A_2A_3m$ is represented by G in view of (4.06).

Now let $G(X_1, X_2, X_3, X_4) = A_1A_2A_3m$. We seek integers x_1, x_2, x_3, x_4 such that $f(x_1, x_2, x_3, x_4) = m$. We take $x_4 = X_4$. By (4.09),

(4.11) $$A_1A_2A_3m = A_3A_2X_1^2 + o_1A_3X_2^2 + o_1o_2A_1X_3^2 + o_1o_2o_3A_1A_2x_4^2,$$

and hence

$$o_1o_2A_1X_3^2 + o_1o_2o_3A_1A_2x_4 \equiv 0 \pmod{A_3}.$$

But, by (4.07),

(4.12) $$o_3A_2 \equiv -[\alpha_{34}^{(2)}]^2 \equiv -s^2 \pmod{A_3},$$

whence

$$o_1o_2A_1(X_3^2 - s^2x_4^2) \equiv 0 \pmod{A_3}.$$

If

(4.14) $$(o_1o_2A_1, A_3) = 1,$$

then

$$(X_3 - sx_4)(X_3 + sx_4) \equiv X_3^2 - s^2x_4^2 \equiv 0 \pmod{A_3},$$

since A_3 is either an odd prime or double such a prime, we have

$$X_3 \equiv sx_4 \quad \text{or} \quad -X_3 \equiv sx_4 \pmod{A_3},$$

whence, replacing s by its value in (4. 12), we get

$$\pm X_3 = A_3x_3 + \alpha_{34}{}^{(2)}x_4$$

with integral x_3. Substituting the resulting value of $X_3{}^2$ into (4. 11) we get

$$(4.15)\quad A_3A_2A_1m = A_3A_2X_1{}^2 + o_1A_3X_2{}^2 + o_1o_2A_1[(A_3x_3 + \alpha_{34}{}^{(2)}x_4)^2 + o_3A_2x_4{}^2].$$

Replacing o_3A_2 by its value in (4. 07), squaring the expression in the parenthesis, combining the similar terms, and dividing both members of (4. 15) by the factor A_3 common to all terms, we get

$$(4.151)\quad A_2A_1m = A_2X_1{}^2 + o_1X_2{}^2 + o_1o_2A_1(\sum_{i,j=3}^{4}\alpha_{ij}{}^{(2)}x_ix_j).$$

This equality, in turn, implies that

$$o_1X_2{}^2 + o_1o_2A_1(\sum_{i,j=3}^{4}\alpha_{ij}{}^{(2)}x_ix_j) \equiv 0 \pmod{A_2}.$$

But, by (4. 04),

$$o_2A_1\alpha_{ij}{}^{(2)} \equiv -\alpha_{2i}{}^{(1)}\alpha_{2j}{}^{(1)} \pmod{A_2},$$

and hence

$$\begin{aligned} o_2A_1\sum_{i,j=3}^{4}\alpha_{ij}{}^{(2)}x_ix_j &= o_2A_1(\alpha_{33}{}^{(2)}x_3{}^2 + 2\alpha_{34}{}^{(2)}x_3x_4 + \alpha_{44}{}^{(2)}x_4{}^2) \\ &\equiv -[(\alpha_{23}{}^{(1)})^2x_3{}^2 + 2\alpha_{23}{}^{(1)}\alpha_{24}{}^{(1)}x_3x_4 + (\alpha_{24}{}^{(1)})^2x_4{}^2] \pmod{A_2} \\ &\equiv -(\alpha_{23}{}^{(1)}x_3 + \alpha_{34}{}^{(1)}x_4)^2 \pmod{A_2}. \end{aligned}$$

Thus, (4. 151) becomes

$$o_1[X_2{}^2 - (\alpha_{23}{}^{(1)}x_3 + \alpha_{24}{}^{(1)}x_4)^2] \equiv 0 \pmod{A_2}.$$

If

$$(4.16)\quad (o_1, A_2) = 1,$$

then

$$\begin{aligned} &(X_2 - \alpha_{23}{}^{(1)}x_3 - \alpha_{24}{}^{(1)}x_4)(X_2 + \alpha_{23}{}^{(1)}x_3 + \alpha_{24}{}^{(1)}x_4) \\ &\qquad = X_2{}^2 - (\alpha_{23}{}^{(1)}x_3 + \alpha_{24}{}^{(1)}x_4)^2 \equiv 0 \pmod{A_2}, \end{aligned}$$

and since A_2 is a prime or a double of a prime, we have

$$X_2 = A_2x_2 + \alpha_{23}{}^{(1)}x_3 + \alpha_{24}{}^{(1)}x_4 \quad \text{or} \quad -X_2 = A_2x_2 + \alpha_{23}{}^{(1)}x_3 + \alpha_{24}{}^{(1)}x_4,$$

where x_2 is an integer. Substituting the resulting value of $X_2{}^2$ into (4. 151), we get

$$(4.17)\quad A_2A_1m = A_2X_1{}^2 + o_1(A_2x_2 + \alpha_{23}{}^{(1)}x_3 + \alpha_{24}{}^{(1)}x_4)^2 + o_1o_2A_1(\sum_{i,j=3}^{4}\alpha_{ij}{}^{(2)}x_ix_j).$$

Replacing $A_1o_2\alpha_{ij}{}^{(2)}$ by their values in (4. 04), squaring the expression in the

parenthesis, combining similar terms, and dividing both members of (4.17) by the factor A_2 common to all terms, we get

$$A_1 m = X_1^2 + o_1 \sum_{i,j=2}^{4} \alpha_{ij}^{(1)} x_i x_j. \tag{4.171}$$

Again, the last equality implies that

$$X_1^2 + o_1 \sum_{i,j=2}^{4} \alpha_{ij}^{(1)} x_i x_j \equiv 0 \pmod{A_1}. \tag{4.172}$$

But, by (4.02), $o_1 \alpha_{ij}^{(1)} \equiv -a_{1i}a_{1j} \pmod{A_1}$ and hence

$$o_1 \sum_{i,j=2}^{4} \alpha_{ij}^{(1)} x_i x_j \equiv -\sum_{i,j=2}^{4} a_{1i}a_{1j}x_i x_j = -(a_{12}x_2 + a_{13}x_3 + a_{14}x_4)^2 \pmod{A_1}.$$

Thus, (4.172) becomes

$$(X_1 - a_{12}x_2 - a_{13}x_3 - a_{14}x_4)(X_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4) \equiv X_1^2 - (a_{12}x_2 + a_{13}x_3 + a_{14}x_4)^2 \equiv 0 \pmod{A_1}.$$

Since A_1 is a prime, we have

$$X_1 \text{ or } -X_1 = A_1 x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4$$

for an integer x_1. Substituting the resulting value of X_1^2 into (4.171), we get

$$A_1 m = (A_1 x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4)^2 + o_1 \sum_{i,j=2}^{4} \alpha_{ij}^{(1)} x_i x_j. \tag{4.173}$$

Replacing $o_1 \alpha_{ij}^{(1)}$ by their values in (4.02), squaring the expression in the parenthesis, combining similar terms and dividing both members of (4.173) by the factor A_1 common to all terms, we get

$$m = \sum_{i,j=1}^{4} a_{ij} x_i x_j$$

with integral $x_1, \cdots, x_4$. Thus m is represented by f.

FORMS OF ODD DETERMINANTS.

5. A set of sufficient conditions in terms of generic characters. We shall now restrict ourselves to the study of properly primitive forms (2.01) of odd determinants. We shall assume that such a form f is already in a canonical form of type (C_p). Then, since in this case A_1, o_1, o_2, o_3 are all odd, the conditions (4.14) and (4.16) of Theorem 4.1 hold in view of the choice of A_1, A_2, A_3. Thus if the associate G of f represents the multiple $A_1A_2A_3m$ of m then f represents m. Consequently should G represent all integers $\geq K$, and

therefore all multiples $A_1A_2A_3m \geq K$ of $A_1A_2A_3$ then f would represent all integers $m \geq K/A_1A_2A_3$. But the form G is of the type considered by Kloosterman (7). We may therefore apply to the form G Kloosterman's conditions 1°-5° assuring representation of all large integers.[9]

We note that A_1 is an odd prime. The same may be assumed in this case of A_2 and A_3 by the proof of Theorem 3. 0. Our choice of A_1, A_2, A_3 implies that 4° and 5° hold. Condition 3° becomes

$$o_1 = 1 \tag{5.1}$$

and, if we write ω_2 for an odd prime factor of o_2, condition 2° becomes

$$(A_2|\omega_2) = (-1|\omega_2) \text{ or } (A_2|\omega_2) = (-o_3|\omega_2), \text{ or both, if } \omega_2^2 \nmid o_2 \text{ and } \omega_2 \nmid o_3; \tag{5.21}$$

$$(A_2|\omega_2) = (-1|\omega_2), \text{ if either (1) } \omega_2^2 | o_2 \text{ or (2) } \omega_2 | o_3; \tag{5.22}$$

$$(-o_3A_2|A_3) = 1, \ (-o_2A_1A_3|A_2) = 1, \ (-A_2|A_1) = 1. \tag{5.23}$$

The condition (5. 23) is satisfied by all forms, for, by virtue of (4. 07), (4. 02) and (4. 04), we have

$$-o_3A_2 \equiv (\alpha_{34}^{(2)})^2 \pmod{A_3}, \quad -A_2 \equiv a_{12}^2 \pmod{A_1},$$
$$-o_2A_1A_3 \equiv (\alpha_{23}^{(1)})^2 \pmod{A_2}.$$

The condition (5. 1) restricts the value of the first invariant o_1 of f. Next, in view of the choice of A_2 and the definition of the generic characters of f, it is clear that the relations (5. 21) and (5. 22) are in fact conditions upon the generic characters of f with respect to the odd prime factors of o_2.

The condition (5. 21) may be modified by virtue of the following considerations. If $(o_3|\omega_2) = -1$, then $(-1|\omega_2)$ and $(-o_3|\omega_2)$ have opposite signs and $(A_2|\omega_2)$ must be equal to one or to the other, and hence at least one (and, of course, only one) of the relations in (5. 21) holds true. If, however, $(o_3|\omega_2) = 1$, then $(-1|\omega_2) = (-o_3|\omega_2)$ and the two relations in (5. 21) coincide and, therefore, either both hold true or both fail according as $(A_2|\omega_2) = (-1|\omega_2)$ or $(A_2|\omega_2) = -(-1|\omega_2)$. Thus (5. 21) may be replaced by

$$(A_2|\omega_2) = (-1|\omega_2) \text{ if } \omega_2^2 \nmid o_2, \ \omega_2 \nmid o_3, \ (o_3|\omega_2) = 1. \tag{5.24}$$

The part[10] $(\mu_a, \mu_b, \mu_c, \mu_d) = (0, 0, 0, 0)$ of the condition 1° is in fact the necessary and sufficient condition in order that a form (1. 1) with odd coefficients a, b, c, d should fail to represent zero properly modulo 8. In view of the choice of A_1, A_2, A_3 and the formulae (4. 02), (4. 05), (4. 08), the form f

[9] (7), Section 4. 6, p. 453.
[10] Kloosterman (7), p. 453.

in (3.01) and its associate G in (4.09) either both represent zero properly modulo 8 or both fail to do so. The above mentioned conditions

$$(5.25) \qquad a \equiv b \equiv c \equiv d \pmod 4 \quad a+b+c+d \equiv 4 \pmod 8$$

as applied to G, yield conditions

$$(5.26) \qquad o_1 \equiv o_3 \pmod 8,\ o_3A_2 \equiv 1 \pmod 4,\ o_2A_1A_3 \equiv 1 \pmod 4.$$

For, (5.251) becomes $A_3A_2 \equiv o_1A_3 \equiv o_1o_2A_1 \equiv o_1o_2o_3A_1A_2 \pmod 4$, and hence $A_2 \equiv o_1$, $A_3 \equiv o_2A_1$, $1 \equiv o_3A_2 \pmod 4$. The first and the third of these last congruences imply $o_1 \equiv o_3 \pmod 4$. Moreover, (5.252) becomes

$$A_3(A_2+o_1) + o_1o_2A_1(1+o_3A_2) \equiv 4 \pmod 8.$$

Since $A_2+o_1 \equiv 2o_1 \equiv 2$ and $1+o_3A_2 \equiv 2 \pmod 4$, each of the two terms above is double an odd integer and the last congruence together with $A_3 \equiv o_2A_1 \pmod 4$ implies $A_3(A_2+o_1) \equiv o_1A_3(1+o_3A_2) \pmod 8$. Therefore $A_2+o_1 \equiv o_1+o_1o_3A_2$ and $1 \equiv o_1o_3 \pmod 8$. It is easily seen that conditions (5.26) imply that G satisfies (5.25).

The conditions (5.262) and (5.263) are equivalent to

$$(5.27) \qquad \Psi = (-1)^{\frac{1}{2}(o_3A_2+1)\cdot\frac{1}{2}(o_2A_1A_3+1)} = -1.$$

Since the ternary form $F(0, X_2, X_3X_4)$ in Section **3**, has invariants $\Omega = o_2$ and $\Delta = o_2A_1$, and since A_3 and A_2 are represented simultaneously by this ternary and its reciprocal we have [11]

$$\Psi \cdot (A_3|o'_3)(A_2|o'_2A_1) = (-1)^{\frac{1}{2}(o_3+1)\cdot\frac{1}{2}(o_2A_1+1)}.$$

Here $o_i = o_i'o_i''^2$, and $o_i''^2$ is the largest square dividing o_i. In view of (4.02),

$$(A_2|A_1) = (-o_1|A_1) = (-1)^{[(A_1-1)/2]\cdot[(o_1+1)/2]}(A_1|o_1'),$$

and therefore

$$(A_2|o'_2A_1) = (A_2|o'_2)(A_1|o'_1)(-1)^{\frac{1}{2}(A_1-1)\frac{1}{2}(o_1+1)}.$$

Since (5.261) implies that $\frac{1}{2}(o_3+1)\frac{1}{2}(o_3A_1+1) + \frac{1}{2}(o_1+1)\frac{1}{2}(A_1-1) \equiv \frac{1}{2}(o_3+1)$ modulo 2, we have

$$(5.28) \qquad (A_3|o'_3)(A_2|o'_2)(A_1|o'_1) = \Psi\cdot(-1)^{\frac{1}{2}(o_3+1)} = -(-1)^{\frac{1}{2}(o_3+1)}$$

in view of (5.27).

We see that conditions (5.26) are equivalent to (5.261) and (5.28). In view of the choice of A_1, A_2, A_3 condition (5.28) is a restriction upon the generic characters of f.

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[11] Smith, vol. 1, p. 470.

If (5.1) holds, then, in view of (5.261), (5.28) becomes

$$(5.29) \qquad (A_3|o'_3)(A_2|o'_2) = 1.$$

Employing the notation of Minkowski,[10] we may state our conclusions in the following form:

THEOREM 5. *Let ϕ be a properly primitive quaternary form of odd determinant and the invariants o_1, o_2, o_3. Let ω_2 be an odd prime factor of o_2. Let finally*

$$(5.1) \qquad o_1 = 1,$$

$$(5.31) \quad (\phi_2|\omega_2) = (-1|\omega_2) \text{ for } \omega_2 \text{ such that } \omega_2^2 \nmid o_2o_3,\ (o_3|\omega_2) = 1,$$

$$(5.32) \quad (\phi_2|\omega_2) = (-1|\omega_2) \text{ for } \omega_2 \text{ such that } \omega_2^2 | o_2o_3,$$

$$(5.33) \quad (\phi_3|o'_3)(\phi_2|o'_2) = -1, \text{ if } o_3 \equiv 1 \pmod 8.$$

Then the form ϕ represents all but a finite number of integers.

The truth of this theorem follows at once from the fact that conditions (5.1), (5.31)-(5.33) imply that (5.1), (5.22)-(5.24) but not (5.26) hold for the associate G of a canonical form f of type (C_p) in the class of ϕ and hence G represents all but a finite number of integers.

6. The necessity of conditions (5.1) and (5.32). We ask now if the conditions (5.1), (5.31), (5.32), (5.33) are necessary in order that ϕ should represent all integers with but a finite number of exceptions. We find at once that (5.1) and (5.32) are necessary. For, should there be an odd prime divisor $\omega_1 > 1$ of o_1, ϕ would not represent integers m such that

$$(6.1) \qquad (m, \omega_1) = 1, \qquad (m|\omega_1) = -(\phi_1|\omega_1).$$

(in view of (4.11)). Next, suppose that (5.32) should fail. Then $(\phi_2|\omega_2) = -(-1|\omega_2)$. First let $\omega_2^2 | o_2$. Then, in view of (4.11), with $o_1 = 1$, and the choice of A_2, A_1, ϕ would not represent integers m such that

$$(6.2) \qquad m = \omega_2 m_1, \qquad (m_1, \omega_2) = 1.$$

Next let $\omega_2^2 \nmid o_2$, but $\omega_2 | o_3$. Then by (4.11), with $o_1 = 1$, ϕ would not represent integers m such that

$$(6.3) \quad m = \omega_2 m_1,\ (m_1, \omega_2) = 1,\ (m_1|\omega_2) = -(\phi_3|\omega_2)(\phi_2|\omega_2)(\bar{o}_2|\omega_2),$$

where $o_2 = \omega_2 \bar{o}_2$.

One sees without any difficulty that there are infinitely many integers of any one of the types (6.1), (6.2), and (6.3).

7. The conditions (5. 31) and (5. 33). The condition (5. 31) differs essentially from (5. 1) and (5. 32). Its failure does not directly imply that there is an infinity of integers not represented, but rather that *if there is one integer not represented, then there is an infinity of such integers.* More precisely: let ϕ, and therefore f, represent an integer $\omega_2^{2}m$ where ω_2 is an odd prime factor of o_2 such that

$$(7.1) \qquad \omega_2^{2} \nmid o_2 o_3, \quad (o_3 | \omega_2) = 1 \quad \text{and} \quad (\phi_2 | \omega_2) = -(-1|\omega_2).$$

Then the condition (5. 31), and hence (5. 24), fails and we have

$$(7.11) \qquad (A_2|\omega_2) = -(-1|\omega_2) \quad \text{and} \quad (A_2|\omega_2) = -(-o_3|\omega_2).$$

Since $A_1A_2A_3\omega_2^{2}m$ is represented by G we have, in view of (4. 11) with $o_1 = 1$,

$$A_3A_2X_1^{2} + A_3X^{2} \equiv 0 \pmod{\omega_2},$$

and, since $(A_3, w_2) = 1$,

$$A_2X_1^{2} + X_2^{2} \equiv 0 \pmod{\omega_2}.$$

Therefore, by (7.11_1) $X_1 \equiv X_2 \equiv 0 \pmod{\omega_2}$. Replacing X_1 and X_2 by $\omega_2X'_1$ and $\omega_2X'_2$ respectively, in (4. 11), and dividing every term of both members of this equality by ω_2, we see that

$$\bar{o}_2A_1X_3^{2} + \bar{o}_2o_3A_1A_2X_4^{2} \equiv 0 \pmod{\omega_2},$$

and, since $(\bar{o}_2A_1, \omega_2) = 1$,

$$X_3^{2} + o_3A_2X_4^{2} \equiv 0 \pmod{\omega_2}.$$

But by (7.11_2), $(-o_3A_2|\omega_2) = -1$, and hence $X_3 \equiv X_4 \equiv 0 \pmod{\omega_2}$. Write $X_3 = \omega_2X'_3$, $X_4 = \omega_2X'_4$. Then, dividing once more every term of both members of (4. 11) by ω_2, we get

$$A_1A_2A_3m = G(X'_1, X'_2, X'_3, X'_4).$$

Thus G represents $A_1A_2A_3m$, and hence f, and therefore ϕ represents m. It follows, therefore, that if a form ϕ should not fulfil condition (5. 31) then if it does not represent an integer m it also does not represent $\omega_2^{2}m$ and in general $\omega_2^{2\kappa}m$ for every integer κ. This proves the above statement in italics. It is seen then that *such a form ϕ either represents all integers or there are infinitely many integers not represented by ϕ.* We shall speak of the set of integers of the form $\omega_2^{2\kappa}m$ as the *tower* $\omega_2^{2\kappa}m$ or the tower generated by ω_2 and m.

In view of the discussion preceding the statement of Theorem 5, the failure of condition (5. 33) implies that if f does not represent an *even* integer

$2m$, then it does not represent the whole tower $2^{2\kappa+1}m$, that is, it does not represent an infinity of integers.

8. Forms representing all large integers and not covered by Theorem 5. Such forms may be divided into two types. Type P_1, for which (5.31) does not hold, and type P_3 for which (5.31) holds true but (5.33) fails to hold.

Forms of type P_3 must represent all even integers. For, if such a form fails to represent an even integer $2m$ then it fails to represent the whole tower $2^{2\kappa+1}m$. In view of Theorem 2.3, the determinant of such a form does not exceed B_2. Thus, there is only a finite number of classes of forms of type P_3.

Forms of type P_1 must represent all integers. For, if such a form fails to represent an integer m then it fails to represent the whole tower $\omega^{2\kappa}m$. Therefore, by Theorem 2.0, the determinant of such a form does not exceed 112.

We shall determine all forms of type P_1. The only odd determinants $D < 112$ which permit such forms are given together with the desired invariants o_2, o_3 ($(o_3|\omega_2) = 1$) by the following table.

(8.01)

D	9	63	25	49
$\omega_2 = o_2$	3	3	5	7
o_3	1	7	1	1

Since every form ϕ under consideration which does not represent 1 also does not represent ${\omega_2}^{2\kappa}$, we need consider only forms which represent 1. Such forms are equivalent to

$$(8.02)\quad \phi = \xi^2 + ax^2 + by^2 + cz^2 + 2ryz + 2sxz + 2txy = \xi^2 + \mu(x, y, z).$$

If ϕ is a form of one of the determinants given in the table (8.01) and if it possesses the indicated invariants o_2 and o_3 then $\mu(x, y, z)$ is a properly primitive form with invariants $\Omega = o_2$ and $\Delta = o_3$ and odd determinant $\Omega^2\Delta = {o_2}^2 o_3 = D$. Since (7.1) holds, we have

$$(8.03)\qquad (\mu|\omega_2) = -(-1|\omega_2)$$

for the generic character $(\mu|\omega_2)$ of μ. For, if we choose a to be any integer prime to ω_2 and properly represented by μ then, in view of (8.02),

$$(\phi_2|\omega_2) = (a|\omega_2) = (\mu|\omega_2).$$

We may assume next that μ in (8.02) is a reduced form and we need to consider all forms (8.02) in which μ is a reduced form of the above mentioned invariants and genus satisfying (8.03). We list such forms in the following table.

D	$\Omega = \omega_2$	Δ		a	b	c	r	s	t	$(\mu\|\omega_2)$	$(-1\|\omega_2)$
9	3	1	$\mu^{(1)}$	1	3	3	0	0	0	1	-1
25	5	1	$\mu^{(2)}$	2	3	5	0	0	-1	-1	1
49	7	1	$\mu^{(3)}$	1	7	7	0	0	0	1	-1
			$\mu^{(4)}$	2	4	7	0	0	-1		
63	3	7	$\mu^{(5)}$	1	3	21	0	0	0	1	-1
			$\mu^{(6)}$	1	6	12	-3	0	0		
			$\mu^{(7)}$	3	3	7	0	0	0		

We write

$$\phi^{(i)} = \xi^2 + \mu^{(i)} \qquad (i = 1, \cdots, 7).$$

Each of the forms $\phi^{(1)}, \cdots, \phi^{(7)}$ satisfies the conditions (5.1), (5.32), but not the condition (5.31). Hence each of these forms either represents all integers or fails to represent an infinite number of integers (Cf. **7**). One sees at once that

$$\phi^{(3)} \neq 3, \quad \phi^{(5)} \neq 6, \quad \phi^{(6)} \neq 3, \quad \phi^{(7)} \neq 2, \tag{8.1}$$

and therefore these forms belong to the second category, i. e., they do not represent an infinity of integers. The form $\phi^{(1)} = \xi^2 + x^2 + 3y^2 + 3z^2$ as is well known,[12] represents all integers. Consider next the form

$$\phi^{(2)}(\xi, x, y, z) = \xi^2 + 2x^2 + 3y^2 + 5z^2 - 2xy.$$

Its ternary section $\psi = \phi(\xi, x, y, 0) = \xi^2 + 2x^2 + 3y^2 - 2xy$ belongs to a genus of one class [13] of determinant 5. Its invariants are $\Omega = 1$, $\Delta = 5$, and its character is

$$(\Psi|5) = (2|5) = -1 = -(-\Omega|5)$$

where Ψ is the reciprocal of ψ. Therefore [14] integers not represented by ψ are

$$5^{2k+1}(5n+1), \quad 5^{2k+1}(5n+4). \tag{8.11}$$

In order to prove that $\phi^{(3)}$ represents all integers we need only prove that it represents all integers of the form (8.11). But these integers are represented by the ternary section $\phi^{(2)}(\xi, x, 0, z) = \xi^2 + 2x^2 + 5z^2$ which represents all integers not of the form [15] $5^{2k+1}(5n+2)$, $5^{2k+1}(5n+3)$ and hence all integers of the form (8.11). Thus $\phi^{(2)}$ represents all integers.

Finally, consider the form

$$\phi^{(4)} = \xi^2 + 2x^2 + 4y^2 + 7z^2 - 2xy.$$

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[12] Ramanujan (14), Dickson (4), pp. 111-113.
[13] Jones (6), Borissow (1).
[14] Ross (17), Lemmas 1-3.
[15] Ramanujan (14), Dickson (4), pp. 111-113.

Since $2\phi^{(4)} = 2\xi^2 + (2x-y)^2 + 7y^2 + 14z^2$, the form $\phi^{(4)}$ will represent all integers if

$$\psi^{(4)} = X^2 + 2\xi^2 + 7y^2 + 14z^2$$

will represent all even integers. The ternary section

$$X^2 + 2\xi^2 + 7y^2 \tag{8.12}$$

represents [16] all even integers $\equiv 0$ or $1 \pmod 3$ which are not of the form

$$7^{2\kappa+1}(14m + R); \qquad R = 10,\ 12,\ \text{and}\ 6. \tag{8.13}$$

If an integer is $\equiv 0$ or $1 \pmod 3$ and is of the form (8.13), we need concern ourselves only with those of type $L = 7(14m + R)$. Since at least one of the integers

$$L - 14 \cdot 3^2 = 7[14(m-1) + (R-4)]$$

or

$$L - 14 \cdot 6^2 = 7[14(m-5) + (R-2)]$$

is not of the form (8.13) and neither one is $\equiv 2 \pmod 3$, one of them is represented by (8.12), and hence L, and therefore $7^{2\kappa}L$, is represented by $\phi^{(4)}$.

Finally, let an integer be of the form $3n + 2$. If $z \not\equiv 0 \pmod 3$ then $3n + 2 - 14z^2 \equiv 3n + 2 - 2 \equiv 0 \pmod 3$. We may again assume that $7^2 \nmid 3n + 2$. If $(7, 3n + 2) = 1$, then $3n + 2 - 14$ is not of the form (8.13) and hence is represented by (8.12). If $7 \mid 3n + 2$, then

$$3n + 2 = 7(14m + M), \qquad M \not\equiv 0 \pmod 7.$$

In this case at least one of the integers

$$7(14m + M) - 14 = 7[14m + (M-2)],$$
$$7(14m + M) - 14 \cdot 2^2 = 7[14m + (M-8)],$$

or

$$7(14m + M) - 14 \cdot 4^2 = 7[14(m-2) + (M-4)]$$

is not of the form (8.13). The only numbers for which the above differences are negative, are ≤ 560 and are easily seen to be represented by $\phi^{(4)}$. Thus $\phi^{(4)}$ represents all integers.

We may now supplement Theorem 5 by

THEOREM 8. *Except for forms given in Theorem 5, there is only a finite number of classes of forms of odd determinants representing all large integers. These consist of a finite number of classes of forms of type P_3 and exactly*

[16] Pall (12).

three classes of type P_1, viz., the three classes containing $\phi^{(1)}$, $\phi^{(2)}$ and $\phi^{(4)}$. The classes to which $\phi^{(1)}$ and $\phi^{(2)}$ belong may be completely described by their generic invariants, for they are the only classes in their respective genera.[17] *This is not true of the class of $\phi^{(4)}$.*

9. One observes that $\phi^{(3)} = \xi^2 + \mu^{(3)}$ and $\phi^{(4)} = \xi^2 + \mu^{(4)}$ (Cf. the table in **8**) do not represent the same large integers even though they belong to the same genus. For, $\phi^{(4)}$ represents all large integers, whereas $\phi^{(3)}$ does not represent integers in the tower $3 \cdot 7^{2\kappa}$. Similarly the forms $\phi^{(5)}$, $\phi^{(6)}$, $\phi^{(7)}$ do not represent the same large integers in view of (8.1). It appears that when a quaternary form f fails to represent zero properly modulo p^α where $\alpha \geq 2$ and p is a prime, then the behavior of f for large integers depends not only upon the values of its generic invariants but also upon its accidental behavior for small values of the variables. For, should f fail to represent a small integer $p^\nu m$, it would not represent the whole tower $mp^{2\kappa+\nu}$. In our case when $p = \omega_2$ then $\nu = 0$ and when $p = 2$ then $\nu = 1$.

The failure of (5.31) or (5.33) implied that our form was not a zero form modulo $\omega_2{}^2$ or modulo 8 respectively.

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THE COMPLETION OF A PROBLEM OF KLOOSTERMAN.*

By GORDON PALL.

1. Introduction. The Euler-Lagrange proof of the theorem that every positive integer is a sum of four squares employed the fact that the form $x^2 + y^2 + z^2 + t^2$ is multiplicative. Hence Liouville [1] [1] examined the multiplicative forms $x^2 + ay^2 + bz^2 + abt^2$, and found the positive integers a, b, for which these forms represent all positive integers. Ramanujan [2] examined in similar fashion the forms

$$f = (a, b, c, d) = ax^2 + by^2 + cz^2 + dt^2, \tag{1}$$

where a, b, c, d are positive integers, and $a \leqq b \leqq c \leqq d$. He found that there are 54 such forms which represent all positive integers; actually he had 55, and Dickson later pointed out his error [3]. More recently, Halmos found the 88 such forms which represent all positive integers with one exception [4].

Ramanujan, in the spirit of the analytic number theory which was then becoming popular, proposed and partly solved the more interesting problem of determining all forms (1) which represent all but a finite number of positive integers. Kloosterman [5] later solved this problem of Ramanujan, save that he was unable to decide whether the four forms

$$(1, 2, 11, 38), \quad (1, 2, 17, 34), \quad (1, 2, 19, 22), \quad (1, 2, 19, 38) \tag{2}$$

represent all positive integers. In **11-13** we shall complete the problem by showing that these four forms do in fact represent all large integers. The technique used for this purpose is based on the fact that quadratic forms in the same genus have rational transformations into one another, which can be employed in particular cases to investigate the numbers represented integrally by the individual forms. Earlier similar attempts by the writer (6) failed because he did not then realize that Kloosterman's asymptotic formula could be adapted to settle the problem for numbers involving a limited power of 2, so that it is only necessary to consider, say, multiples of 4.

However, the most interesting feature of this paper consists in the elegant formulation of the conditions for a form to represent all large integers. This

* Received July 17, 1945.

[1] Numbers in square brackets refer to the references at the end of the paper.

formulation we owe largely to reading a manuscript of Arnold E. Ross, in which he considers the extension of Kloosterman's results to non-diagonal forms. Also, whereas much earlier work on quadratic forms is complicated by many cases involving the numerous invariants of the forms, we obtain our results here directly and simply by appealing to the arithmetical properties of the forms themselves. Although our Theorems 1, 2, and 4 are true, precisely as stated, for non-diagonal forms as well, this will not be proved here.

2. The pertinent properties of f. If f is to represent all large integers, then evidently the congruence $f \equiv n \pmod{k}$ must be solvable in integers x, y, z, t, for every pair of integers n and k, $k \neq 0$. If p is any prime we shall say that f *is p-adically universal*, when

$$ax^2 + by^2 + cz^2 + dt^2 \equiv n \pmod{p^r}, \tag{3}$$

is solvable in integers x, y, z, and t, for every n and r, $r \geqq 0$. Hence a necessary condition for f to represent all large integers is that f be p-adically universal for every p.

We shall later (Lemma 1) give precise criteria for such universality, as also (Lemma 2) for p-adic representation of zero.

We shall say that f *fails to represent zero p-adically*, if

$$ax^2 + by^2 + cz^2 + dt^2 \equiv 0 \pmod{p^r} \tag{4}$$

implies, for some r, that $x \equiv y \equiv z \equiv t \equiv 0 \pmod{p}$. If, however, (4) is solvable for every positive integer r in integers x, y, z, t not all divisible by p, we say that f *represents zero p-adically*. The connection with our problem is this. If f fails to represent zero p-adically for some p, suppose for a certain s, that $f \equiv 0 \pmod{p^s}$ implies $x \equiv y \equiv z \equiv t \equiv 0 \pmod{p}$. Then the number of representations of $p^{s+2k}n$ is the same as that of $p^s n$ in f, for every positive integer k. Hence *if* f does not represent one integer of the form $p^s n$, f does not represent infinitely many integers. And in any case the number of representations of the *large* number p^{s+2k} is bounded, so that, in the asymptotic formula soon to be encountered, $\chi(p, p^{s+2k})$ is near zero when k is large.

3. The principal theorems. Kloosterman's results are expressed in rather complicated fashion in terms of conditions on the coefficients of f. Interpreting his results by means of the notions of **2**, and using Lemmas 1 and 2, we get the following two theorems.

THEOREM 1. *If* (a) *f is p-adically universal for every p, and* (b) *f represents zero p-adically for every p, then f represents all sufficiently large integers.*

It will be observed that (a) is a necessary, and (b) is not a necessary condition.

THEOREM 2. *Of the forms f which are p-adically universal for every p, there are only a finite number of forms which fail to represent zero p-adically for some prime p_1, and yet represent all large integers.*

We shall in fact prove

THEOREM 3. *There are precisely* 199 *forms which fail to represent zero p-adically for some p_1, and yet represent all large integers. They are as follows, the first two having $p_1 = 3$ and 5, the others $p_1 = 2$:*

(A) $(1, 1, 3, 3)$;

(B) $(1, 2, 5, 10)$;

(C) $(1, 1, 5, 5)$, *and* $(1, 1, 1, t)$ $(t = 1, 9, 17, 25)$;

(D) $(1, 4, 5, 5)$, $(1, 1, 5, 20)$, $(1, 1, 4, t)$, $(1, 1, 1, 4t)$;

(E) $(1, 1, 10, 10)$, $(2, 2, 5, 5)$, $(1, r, 2, 2t)$, $(1, 17, 2, 2s)$, *and* $(1, 25, 2, 2r)$, $(r = 1, 9;\ s = 1, 9, 17)$;

(F) $(1, 1, 10, 40)$, $(5, 5, 2, 8)$, $(1, r, 2, 8t)$, $(1, r, 8, 2t)$, $(1, 17, 2, 8s)$, $(1, 17, 8, 2s)$, $(1, 25, 2, 8r)$, $(1, 25, 8, 2r)$;

(G) $(1, 4, 10, 10)$, $(2, 2, 5, 20)$, $(1, 4r, 2, 2t)$, $(4, r, 2, 2t)$, $(1, 68, 2, 2s)$, $(4, 17, 2, 2s)$, $(1, 100, 2, 2r)$, $(4, 25, 2, 2r)$;

(H) $(1, 4, 10, 40)$, $(2, 8, 5, 20)$, $(1, 4r, 2, 8t)$, $(1, 4r, 8, 2t)$, $(4, r, 2, 8t)$, $(4, r, 8, 2t)$, $(1, 68, 2, 8s)$, $(1, 68, 8, 2s)$, $(4, 17, 2, 8s)$, $(4, 17, 8, 2s)$, $(1, 100, 2, 8r)$, $(1, 100, 8, 2r)$, $(4, 25, 2, 8r)$, $(4, 25, 8, 2r)$;

(I) $(1, u, 2, 2v)$, $u, v = 3, 11,$ *and* 19;

(J) $(1, 4u, 2, 2v)$, $(4, u, 2, 2v)$;

(K) $(1, u, 2, 8v)$, $(1, u, 8, 2v)$;

(L) $(1, 4u, 2, 8v)$, $(1, 4u, 8, 2v)$, $(4, u, 2, 8v)$, $(4, u, 8, 2v)$.

Furthermore, we shall give in **4** a surprisingly easy proof of a theorem of which Theorem 1 is obviously a corollary:

THEOREM 4. *Let n denote any integer such that $f \equiv n \pmod{k}$ is solvable for every modulus k. For each prime p such that f fails to represent zero p-adically, impose an upper bound to the power of p in n. Then f represents every sufficiently large n (> 0) thus restricted.*

3a. Modification of the asymptotic formula. Kloosterman's formula [7] for the number of representations $f(n)$ of n by f is

$$f(n) = \frac{\pi^2}{(abcd)^{1/2}} nS(n) + O(n^{17/18+\epsilon}). \tag{5}$$

To prove that $f(n) > 0$ for n large, it suffices to show that

$$S(n) > K/\log\log n, \tag{6}$$

where K is a positive constant depending only on f and the positive number ϵ. Kloosterman expresses $S(n)$ as a product over all primes p, namely

$$S(n) = \prod_p \chi(p), \tag{7}$$

$$\chi(p) = \chi(p, n) = 1 + A(p) + A(p^2) + \cdots, \tag{8}$$

$$p^{4r}A(p^r) = \sum_h \sum_{x,y,z,t} \exp[2\pi i h(ax^2 + by^2 + cz^2 + dt^2 - n)/p^r], \tag{9}$$

where x, y, z, t range over all residues mod p^r, and h over all such residues prime to p. To put $\chi(p)$ into a more significant form, note that if $r \geqq 1$, the right member of (9) is

$$\begin{aligned} &\sum_{h,x,y,z,t \bmod p^r} \exp[2\pi i h(ax^2 + by^2 + cz^2 + dt^2 - n)/p^r] \\ &\quad - p^4 \sum_{h_1,x,y,z,t \bmod p^{r-1}} \exp[2\pi i h_1(ax^2 + by^2 + cz^2 + dt^2 - n)/p^{r-1}] \\ &= p^r f(n, p^r) - p^{r+3} f(n, p^{r-1}), \end{aligned}$$

where $f(n, k)$ denotes the number of solutions of

$$f(x, y, z, t) \equiv n \pmod{k}. \tag{10}$$

Hence

$$\chi(p) = \lim_{r\to\infty} p^{-3r} f(n, p^r), \tag{11}$$

where it should be observed that if p^δ is the precise power of p in the determinant of f, $p^{-3r}f(n, p^r)$ is independent of r if $r \geqq \delta + 3$ [8]. This interesting form of expression for $\chi(p)$ will be found in a paper by Tartakowsky [9], and in the work of Siegel.

We note that it is easy to prove by (11) that

$$\text{if } p \nmid 2abcd, \qquad \chi(p) = (1 - \epsilon_1 p^{-2}) \sum_{j=0}^{\nu} \epsilon_1^j p^{-j}, \tag{12}$$

where $\epsilon_1 = (abcd|p)$, $n = p^\nu n_1$, n_1 prime to p.

A sketch of Kloosterman's application of (5) is now in order. He proves easily that the product of the $\chi(p)$ over all primes save the finite number dividing $2abcd$, exceeds $K/\log\log n$ [10]. He then narrows the problem to forms whose coefficients satisfy certain conditions. Comparison with our Lemma 1 will show that these are the conditions for f to be universal for every p. Next, confining himself to such forms, he shows that for primes of a special kind, which by our Lemma 2 we now recognize as those for which f fails to represent zero p-adically, $\chi(p)$ is so near zero when n is divisible by a high power of p that (6) will not hold. This is to be expected from the last observation of **2**.

4. Proof of Theorem 4. We shall formulate the proof so as to apply to any m-ary quadratic form f for which a formula like (5) holds; $\chi(p)$ is then $\lim p^{-(m-1)r} f(n, p^r)$. Except possibly when $p = 2$ (see below) any f can be expressed modulo p^r as a sum of terms $p^{\alpha_i} a_i x_i^2$, say

$$f \equiv p^{\alpha_1} a_1 x_1^2 + \cdots + p^{\alpha_m} a_m x_m^2, \tag{13}$$

$$0 \leqq \alpha_1 \leqq \alpha_2 \leqq \cdots \leqq \alpha_m, \qquad a_1 \cdots a_m \text{ prime to } p.$$

To prove the theorem it is evidently sufficient to show that $\chi(p)$ is bounded away from zero for all large n, for each prime p dividing the determinant of f, and for the prime $p = 2$.

Case 1. Suppose n to be such that $f \equiv n \pmod{p^{\alpha_m+3}}$ is solvable with some x_i prime to p. We shall prove that $\chi(p) \geqq p^{-(m-1)(\alpha_m+3)}$. For, proceeding by induction, suppose $r \geqq \alpha_m + 3$ and that $f \equiv n \pmod{p^r}$ is solvable with, say, x_3 prime to p. The residues $x_1, x_2, x_4, \cdots, x_m \pmod{p^r}$ expand into p^{m-1} sets of residues mod p^{r+1}. For each of these we can choose h so that if x_3 is replaced by $x_3 + p^{r-\alpha_3} h$ if $p > 2$, or by $x_3 + 2^{r-\alpha_3-1} h$ if $p = 2$, then $f \equiv n \pmod{p^{r+1}}$ [11]. By repetitions of this process we see that $f \equiv n \pmod{p^r}$ has at least $p^{(m-1)(r-\alpha_m-3)}$ solutions, if $r \geqq \alpha_m + 3$. Hence $\chi(p) \geqq p^{-(m-1)(\alpha_m+3)}$.

Case 2. Let the power p^ν of p in n be bounded. Then if $f \equiv n \pmod{p^r}$ has p^ρ dividing every x_i but not $p^{\rho+1}$, then $2\rho \leqq \nu$ and the number of solutions of $f \equiv n \pmod{p^r}$ is $p^{m\rho}$ times the number of solutions of $f \equiv n/p^{2\rho} \pmod{p^{r-2\rho}}$. Hence by Case 1,

$$f(n, p^r) \geqq p^{m\rho} p^{(m-1)(r-2\rho-\alpha_m-3)}, \qquad \chi(p) \geqq p^{-\frac{1}{2}(m-2)\nu-(m-1)(\alpha_m+3)}.$$

Case 3. Finally, let f represent zero p-adically. The fact that $\chi(p)$ is

bounded away from zero follows from Case 2 if $p^{a_{m+3}} \nmid n$. Let then $p^{a_{m+3}} | n$ Now, $f \equiv 0 \pmod{p^{a_{m+3}}}$ is solvable with an x_i prime to p. Hence $f \equiv n \pmod{p^{a_{m+3}}}$ is similarly solvable, and Case 1 applies.

Although not needed in this article the remaining case with $p = 2$ will be resolved. We then have $f \equiv 2^a(jx_1^2 + x_1x_2 + jx_2^2) +$ terms in other variables $\pmod{2^r}$, and $j = 0$ or 1. Hence, if $f \equiv n \pmod{2^r}$ is solvable with (say) x_1 odd, and $s \geqq \alpha + 1$, we can replace x_1 by $x_1 + 2^{s-a}h$ and x_2 by $x_2 + 2^{s-a}k$, and obtain $f \equiv n \pmod{2^{s+1}}$ if $2^sQ + 2^s(hx_2 + kx_1) \equiv 0 \pmod{2^{s+1}}$, where Q denotes an integer; i. e. with arbitrary h and unique k mod 2; thus again there are 2^{m-1} times as many solutions of $f \equiv n \pmod{2^{s+1}}$ as of $f \equiv n \pmod{2^s}$, with *any* x_i odd; and $\chi(2) \geqq 2^{-(m-1)(a+1)}$.

This completes the proof of Theorem 4, hence of Theorem 1.

5. The criteria for *p*-adic universality and representation of zero. For any prime p we can number the variables so that (13) holds, with $m = 4$. Detailed proofs of the following three lemmas will be found in a forthcoming book by the author. Since their verification is not difficult we leave it to the reader, noting merely that the easiest proof of Lemma 2 amounts to a direct application of the conditions of Hasse [12] for p-adic representation of zero in rationals x_i.

LEMMA 1. *If $p > 2$, f is p-adically universal if and only if*

(14) $\alpha_1 = \alpha_2 = 0$; *and either* $\alpha_3 = 0$ *or* $(-a_1a_2|p) = 1$ *or* $\alpha_3 = \alpha_4 = 1$.

If $p = 2$, the necessary and sufficient condition is:

(15) $\alpha_1 = 0$, $\alpha_2 = 0$ *or* 1, $\alpha_3 - \alpha_2 = 0$ *or* 1;

(16) *if* $\alpha_2 = 0$, *then either* $\alpha_4 - \alpha_3 \leqq 2$ *or* $c^*_2 = 1$;

(16) *if* $\alpha_2 = 1$, *then either* $\alpha_4 - \alpha_3 \leqq 1$ *or* $c^*_2 = 1$.

Here c^*_2 denotes the unit

$$c^*_2 = (2|a_1a_2)^{a_3}(2|a_1a_3)^{a_2}(-1)^{\frac{1}{2}(a_1a_2+1)\cdot\frac{1}{2}(a_1a_3+1)}.$$

We may observe that c^*_2 is the invariant Ψ of Smith [13], or the invariant c_2 of Hasse, for the ternary form $g = a_1x_1^2 + 2^{a_2}a_2x_2^2 + 2^{a_3}a_3x_3^2$. It has the property that if $c^*_2 = 1$, then g represents zero 2-adically, and f does likewise.

LEMMA 2. *If $p > 2$ and* (14) *holds, then f fails to represent zero p-adically, if and only if*

$$(17) \quad \alpha_1 = \alpha_2 = 0, \ \alpha_3 = \alpha_4 = 1, \ (-a_1a_2|p) = -1 = (-a_3a_4|p).$$

If $p = 2$ and (15) *holds, then f fails to represent zero p-adically if and only if $a_1a_2a_3a_4 \equiv 1 \pmod 8$ and any of the following cases* (18)-(23) *holds:*

$$(18) \quad \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0, \quad a_1 \equiv a_2 \equiv a_3 \equiv a_4 \pmod 4;$$

(19) $\alpha_1 = \alpha_2 = 0, \ \alpha_3 = \alpha_4 = 1,$ *and*

either (i) $a_1 \equiv a_2, \ a_3 \equiv a_4 \pmod 8, \ a_1 \equiv a_3 \pmod 4,$

or (ii) $a_1 \equiv 5a_2, \ a_3 \equiv 5a_4 \pmod 8, \ a_1 \equiv -a_3 \pmod 4,$

or (*iii*) $a_1 \equiv 3a_2, \ a_3 \equiv 3a_4 \pmod 8;$

$$(20) \quad \alpha_1 = \alpha_2 = \alpha_3 = 0, \ \alpha_4 = 2, \ a_1 \equiv a_2 \equiv a_3 \equiv a_4 \pmod 4;$$

(21) $\alpha_1 = \alpha_2 = 0, \ \alpha_3 = 1, \ \alpha_4 = 3,$ *and three cases as in* (19);

(22) $\alpha_1 = 0, \ \alpha_2 = \alpha_3 = 1, \ \alpha_4 = 2,$ *and the three cases of* (19) *with subscripts in the order* 1423;

(23) $\alpha_1 = 0, \ \alpha_2 = 1, \ \alpha_3 = 2, \ \alpha_4 = 3,$ *and the three cases of* (19) *with subscripts in the order* 1324.

The least index s for which, in the cases of Lemma 2, $f \equiv 0 \pmod{p^s}$ implies that all x_i are divisible by p, is given as follows.

LEMMA 3. *When* (17) *holds, $f \equiv 0 \pmod{p^2}$ implies that p divides every x_i. If $p = 2$ and $a_1a_2a_3a_4 \equiv 1 \pmod 8$, then $f \equiv 0 \pmod{2^s}$ implies every x_i even, as follows: when $s = 3$ in* (18), *when $s = 4$ in* (19), *when $s = 5$ in* (20), *when $s = 6$ in* (21)-(23).

6. The forms which fail to represent zero p-adically for an odd p, and yet represent all large integers. By Lemma 3, if f does not represent n it does not represent $p^{2k}n$. Hence such forms f must represent all integers. By (17), f has the form

$$a_1x_1^2 + a_2x_2^2 + p(a_3x_3^2 + a_4x_4^2), \qquad (-a_1a_2|p) = -1 = (-a_3a_4|p),$$

and we can suppose $a_1 \leqq a_2, \ a_3 \leqq a_4, \ p \geqq 3$. In order that f shall represent 1 and 2, $a_1 = 1$ and $a_2 = 1$ or 2. If $p \geqq 7$, f cannot represent 6. If $p = 3$, $(-a_1a_2|p) = -1$ implies that $a_2 = 1$; then if f represents 3,

$a_3 = 1$ and $a_4 \equiv 1 \pmod 3$; if f represents 6, $a_4 = 1$. There remains form (A) of Theorem 3, which is known to represent all positive integers. If $p = 5$ then $a_2 = 2$ in order that $(-a_1a_2|p) = -1$; $a_3 = 1$ if f represents 5, and $a_4 \equiv 2$ or $3 \pmod 5$; $a_4 = 2$ if f represents 10. We thus obtain the form (B), which was overlooked by Kloosterman, and represents all positive integers.

7. The forms satisfying (18) which represent all large integers. By Lemma 3 such forms must represent all even positive integers. Let

$$f = (a_1, a_2, a_3, a_4), \quad a_1 \equiv a_2 \equiv a_3 \equiv a_4 \pmod 4, \quad a_1 \cdots a_4 \equiv 1 \pmod 8,$$
$$a_1 \leqq a_2 \leqq a_3 \leqq a_4.$$

If $a_1 > 1$, or if $a_1 = 1$ and $a_2 \geqq 5$, then $f \neq 2$. If $a_1 = a_2 = 1$ and $a_3 \geqq 9$ $f \neq 6$. If $a_1 = a_2 = 1$, $a_3 = 5$, $a_4 \geqq 9$, then $f \neq 12$. If $a_1 = a_2 = a_3 = 1$ and $a_4 \geqq 33$, then $f \neq 28$. There remain forms (C), treated in **10**.

8. The forms of type (20). We need only consider the forms (C) with a coefficient multiplied by 4, thus getting the nine distinct forms (D). By Lemma 3 it suffices to prove that these represent all multiples of 8. Since the forms (C) represent all large integers, their products by 4 represent all large multiples of 4. Hence the forms (D) represent all large multiples of 4; but none of these forms can fail to represent a number $8n$, since it would then not represent the large numbers $4^k \cdot 8n$. Hence the forms (D) represent all large integers.

9. Further cases in Lemma 2. Next, consider case (19), (i). We use ? to indicate that no multiple of 4 less than 200 is not represented:

$(1, 1, 2, 2t)$?; $(1, 1, 2, \geqq 66) \neq 56$; $(1, 1, 10, 10)$?;
$(1, 1, 10, \geqq 26) \neq 24$; $(1, 1, \geqq 18, \geqq 18) \neq 12$; $(1, 9, 2, 2t)$?;
$(1, 9, 2, \geqq 66) \neq 56$; $(1, \geqq 9, \geqq 10, \geqq 10) \neq 8$;
$(1, 17, 2, 2s)$? if $s = 1, 9, 17$; $(1, 17, 2, \geqq 50) \neq 40$;
$(1, 25, 2, 2r)$? if $r = 1, 9$; $(1, 25, 2, \geqq 34) \neq 20$;
$(1, \geqq 33, 2, \geqq 2) \neq 28$; $(\geqq 3, \geqq 3, \geqq 2, \geqq 6) \neq 4$;
$(5, 5, 2, 2)$?; $(\geqq 5, > 5, 2, 2) \neq 12$.

There remain the 15 forms (E), treated in **10-12**.

Case (21) (i) is got by multiplying one of the even coefficients in the preceding case by 4, yielding the 24 distinct forms (F). To represent all large numbers they need only represent all multiples of 16, and the fact that they do so follows, much as in **8**, from their connection with (E).

Cases (i) of (22) and (23) lead similarly to the $24 + 39$ forms (G) and (H), which represent all large numbers.

Next, take the case of (19) (ii):

$$(1, \geqq 5, \geqq 6, \geqq 14) \neq 8; \qquad (\geqq 3, \geqq 7, \geqq 2, \geqq 10) \neq 4.$$

Hence no such form represents all large integers. The same conclusion follows for cases (ii) of (21)-(23).

Lastly, consider cases (iii). By (19) we have: the nine forms (I) to be treated in **10-13**;

$$(1, u, 2, \geqq 54) \neq 40; \quad (1, \geqq 3, \geqq 6, \geqq 10) \neq 8;$$
$$(1, \geqq 27, 2, \geqq 6) \neq 20; \quad (\geqq 3, \geqq 3, \geqq 2, \geqq 6) \neq 4.$$

Extending these to (21)-(23) we get the 72 forms (J), (K), and (L).

10. The method of ternary sections. To complete the proof of Theorem 3 we need only prove that the forms in (A), (B), (C), (E), (I) represent all large integers. The investigation of such problems is usually made to depend on a knowledge of the numbers represented by a ternary section, obtained by putting one of the variables equal to zero. We know the numbers represented by a genus of ternary quadratic forms, and (with a few exceptions) it is only when a ternary form belongs to a genus of one class that we can tell precisely what numbers it represents. For example, the forms $(1, 1, 3)$, $(1, 2, 5)$, $(1, 1, 5)$, $(1, 1, 1)$, $(1, 1, 2)$, $(1, 2, 2)$, $(1, 2, 3)$, $(1, 2, 6)$, (needed in (A), (B), (C), (E), and (I)), are in genera of one class, and so are known to represent all positive integers except those of the respective forms $3^{2k}(9q + 6)$, $5^{2k}(25q + 10 \text{ or } 15)$, $4^k(8q + 3)$, $4^k(8q + 7)$, $4^k(16q + 14)$, $4^k(8q + 7)$, $4^k(16q + 10)$, and $4^k(8q + 5)$. The forms $(1, 1, 10)$, $(2, 2, 5)$, $(1, 2, 9)$, and $(1, 2, 18)$ (needed in (E)) are not in genera of one class. However, $(1, 1, 10)$ represents $2n$ if $(1, 1, 5)$ represents n; $(1, 2, 9)$ represents $2n$ if $g_0 = y^2 + 2x^2 - 2xz + 5x^2$ (in a genus of one class, representing all $\neq 4^k(8q + 7)$) represents n; $(2, 2, 5)$ represents $4n$ if $(1, 1, 5)$ represents n; $(1, 2, 18)$ represents $4n$ if g_0 represents n. Only the forms in (2) and $(1, 2, 11, 22)$ fail to succumb, by means of these facts, to the following treatment which we illustrate by the form $(1, 25, 2, 18)$.

The form $(1, 25, 2, 18)$ must be shown to represent all multiples of 4. Now $(1, 2, 18)$ represents all multiples of 4 except $4^{k+1}(8q + 7)$. Also, $4^{k+1}(8q + 7) - 4^{k+1} \cdot 25 = 4^{k+1}(8q - 18)$, and this is represented by $(1, 2, 18)$ unless $q = 0$, 1, or 2. Finally, $(1, 25, 2, 18)$ is verified as representing 28, 60, and 92.

11. The forms (1, 2, 11, 22), (1, 2, 11, 38), and (1, 2, 19, 22). The form $(1, 2, 11)$ represents $2n$ if $g_1 = y^2 + 2x^2 - 2xz + 6z^2$ represents n. Now g_1 is in a genus of one class and represents all positive integers not of the form $4^k(8q + 5)$. Hence $(1, 2, 11)$ represents all evens $\neq 4^k(16q + 10)$. Also, $4^{k+1}(16q + 10) - 22 \cdot 4^k = 4^k(64q + 18)$, which is represented by $(1, 2, 11)$. Again, $4^{k+1}(16q + 10) - 38 \cdot 4^k = 4^k(64q + 2)$, also represented by $(1, 2, 11)$. Hence the forms $(1, 2, 11, 22)$ and $(1, 2, 11, 38)$ represent all large multiples of 4; that they represent all large numbers not divisible by 4 follows from Theorem 4. Similarly, $(1, 2, 22)$ represents every $4n \neq 4^{k+1}(8q + 5)$; hence $(1, 2, 22, 19)$ represents every $4n$.

12. The form (1, 2, 17, 34). The form $x^2 + 2y^2 + 17z^2$ represents $2n$ if $g_2 = y^2 + 2x^2 - 2xz + 9z^2$ represents n. Now g_2 is not in a genus of one class, but is in a genus of two classes, the other class containing the form $h_2 = x^2 + y^2 + 17z^2$. Together, g_2 and h_2 represent all positive integers $\neq 4^k(8q + 7)$. However, the identity

$$(24) \quad x^2 + y^2 + 17z^2 = x^2 + 2(3z)^2 - 2(3z)(y + z)/3 + 9[(y + z)/3]^2$$

shows that if n is represented in h_2 with either $y + z$ or $x + z$ divisible by 3, then n is represented also in g_2. This can be arranged unless $x^2 \equiv y^2 \not\equiv z^2 \pmod 3$, whence $n \equiv 2 \pmod 3$. Thus:

g_2 represents every $3s$ and $3s + 1$ not of the form $4^k(8q + 7)$.

Hence $(1, 2, 17)$ represents every $6s$ and $6s + 2$ not of the form $4^k(16q + 14)$. In view of Theorem 4 we need consider only multiples of 4. If $6s$ or $6s + 2 = 4^{k+1}(16q + 14)$, then $6s - 34 \cdot 4^k = 4^k(64q + 22) \equiv 2 \pmod 6$, or $6s + 2 - 34 \cdot 9 \cdot 4^k = 4^k(64q - 250) \equiv 2 \pmod 6$, and thus is represented in $(1, 2, 17)$; except, in the last case when $q = 0$ or 3. But $(1, 2, 17, 34)$ represents 56 and 248.

Finally, there remains $6s + 4 = 4(3s_1 + 1)$, say. Subtracting 34 we get $12s_1 - 30$, which has the form $4^k(16q + 14)$ only if $s_1 = 4s_2 + 1$, whence $4(3s_1 + 1) = 4^2(3s_2 + 1)$. We proceed by induction. If $4^h(3s_h + 1) - 34 \cdot 2^{2h-2}$ or $4^{h-1}(12s_h - 30)$ is of the excluded form, then $s_h = 4s_{h+1} + 1$; and so on. If $4^{h-1}(12s_h - 30)$ is negative, s_h is 0, 1, or 2, and $4^h(3s_h + 1)$ is 4^h, 4^{h+1}, or $4^h \cdot 7$, all of which are evidently represented by $(1, 2, 17)$ if $h \geqq 1$.

13. The form (1, 2, 19, 38). The form $(1, 2, 19)$ represents $2n$ if $g_3 = x^2 + 2y^2 + 2yz + 10z^2$ represents n. Also, g_3 and $h_3 = 2x^2 + 2y^2 + 7z^2 + 2yz + 2zx + 2xy$ constitute a genus, representing all positive integers

$\neq 4^k(8q+5)$. This time, transformations of denominator 3, as in (24), do not suffice, and we use denominator 5. The transformations with the matrices $T/5$, where

$$T = \begin{pmatrix} 6 & 4 & -3 \\ 1 & 4 & 7 \\ 1 & -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 6 & 2 & 9 \\ 1 & -3 & -6 \\ 1 & 2 & -1 \end{pmatrix}, \quad \begin{pmatrix} 4 & -2 & 7 \\ -4 & -3 & 3 \\ 1 & 2 & 3 \end{pmatrix},$$

replace g_3 by h_3. Hence if $n = 2x^2+2y^2+7z^2+2yz+2zx+2xy$ in integers x, y, z, then g_3 also represents n if any of the following congruences holds:

$$x-y+2z \equiv 0, \quad x+2y-z \equiv 0, \quad x+2y-2z \equiv 0,$$
$$x-y-2z \equiv 0, \quad 2x+y-z \equiv 0, \quad 2x+y-2z \equiv 0, \pmod 5,$$

the last three being got by interchanging x and y. Introducing $X = y+z$, $Y = z+x$, $Z = x+y$, we see that these conditions reduce to

(25) $\quad 2X \equiv Y$ or Z, or $2Y \equiv Z$ or X, or $2Z \equiv X$ or $Y \pmod 5$.

If $n \equiv 0, 1$, or $4 \pmod 5$, then $h_3 = n$ implies $X^2+Y^2+Z^2 \equiv 0, 1$, or 4, whence (X, Y, Z) is a permutation of the following residues mod 5: $(0,0,0)$, $(0,0,\pm 1)$, $(0,0,\pm 2)$, $(0,\pm 1,\pm 2)$, $(\pm 1,\pm 1,\pm 2)$, $(\pm 1,\pm 2,\pm 2)$. In all cases, (25) is seen to hold (e. g. $2\cdot 2 \equiv -1$, $2\cdot 1 \equiv 2$). Hence:

g_3 represents every $5s+0, 1, 4$ not of the form $4^k(8q+5)$;

and $(1, 2, 19)$ represents every $10s+0, 2, 8$ not of the form $4^k(16q+10)$.

If $10s = 4^{k+1}(16q+10)$, then $10s - 38\cdot 4^k = 4^k(64q+2)$ and is represented by $(1, 2, 19)$. If $10s+2 = 4^{k+1}(16q+10)$, then according as k is odd or even, $10s+2-38\cdot 4^k = 4^k(64q+2) \equiv 0 \pmod{10}$, or $10s+2-38\cdot 4^{k+1} = 4^{k+2}(4q-7) \equiv 0 \pmod{10}$; in the last case q cannot be 0 or 1. Similar results hold if $10s+8 = 4^{k+1}(16q+10)$ with k even or odd.

Finally, we have $4(5s_1 \pm 1)$ to consider. Now $4(5s_1 \pm 1) - 38 = 4^k(16q+10)$ only if $s_1 \equiv \mp 1 \pmod 4$ respectively. Then $4(5s_1 \pm 1) = 4^2(5s_2 \mp 1)$. We proceed by induction and have $4^h(5s_h \pm 1) - 38\cdot 4^{h-1} = 4^{h-1}(20s_h \pm 4 - 38)$ of the form $4^k(16q+10)$ only if $s_h = 4s_{h+1} \mp 1$; and so on. If $20s_h \pm 4 - 38$ is negative, then s_h is 0, 1, or 2; and we see that $(1, 2, 19, 38)$ represents 4, 16, 24, 36, and 44.

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AN EXTENSION OF A PROBLEM OF KLOOSTERMAN.*

By Arnold E. Ross and Gordon Pall.

1. Introduction. In the previous article [1],[1] Kloosterman's problem of determining all the positive quaternary forms $ax^2 + by^2 + cz^2 + dt^2$ which represent all large integers was completely solved. Ross [2] has proved the following lemma, which makes it possible to extend the solution to cover the general positive integral quaternary form $\Sigma a_{ij}x_ix_j$ in which the a_{ii} and $2a_{ij}$ $(i, j = 1, \cdots, 4)$ are integers.

Lemma 1. *The determinant of any positive integral quaternary quadratic form which represents all positive integers cannot exceed a certain constant R_1.*

We make use also of an extension to a general positive m-ary form $(m \geqq 4)$ of Kloosterman's asymptotic formula [3] for the number of representations of n. Such a formula has been given by W. Tartakowsky [4], who used the Hardy-Littlewood method. It is interesting to see (**2**) how easily this extension can be made directly from Kloosterman's special case, by induction, making use of Ross's [5] technique of employing primes represented by a primitive form, and the fact that every primitive form in two or more variables represents infinitely many primes.

References to the preceding article will be prefixed by P. The definitions of the terms "f is universal for p," and "f represents zero p-adically," given near $P(3)$ and $P(4)$ evidently extend to any integral forms f. Once the asymptotic formula (1) is established, Lemma 2 and the proof of Theorem $P1$ yield

Theorem 1. *Let f be any positive integral m-ary quadratic form, $m \geqq 4$. Let n be such that $f(x_1, \cdots, x_m) \equiv n \bmod p^r$ is solvable for every p and r. For each prime p (if any) such that f fails to represent zero p-adically impose an upper bound to the power of p which may divide n. Then f represents every such n sufficiently large.*

It should be observed that if $m \geqq 5$, every f represents zero p-adically for every $p \mid 6$. Hence we can conclude (as Tartakowsky did) that all forms in a

* Received July 17, 1945.

[1] Numbers in square brackets refer to the references at the end of the paper.

genus of positive quadratic forms in five or more variables represent the same sufficiently large numbers. If $m = 4$, f can fail to represent zero p-adically only for a few primes appearing in its determinant (the conditions [6] are easily applied and are given partly in our Lemma 4).

It follows that if f is universal for every p, and represents zero p-adically for every p, then if $m \geqq 4$, f represents every sufficiently large positive integer n. In **4** we shall, using Lemma 1, obtain

THEOREM 2. *The number of integral positive quaternary quadratic forms which represent all large numbers, and yet fail to represent zero p-adically for some prime p_1, is finite. In fact the determinant of any such form cannot exceed R_1 if $p_1 > 2$, $256R_1$ if $p_1 = 2$.*

Theorem 1 does not extend to $m = 2$, since binary classes in the same genus, which are neither properly nor improperly equivalent, do not represent the same primes 7. The theorem does not extend without modification to $m = 3$, as was shown by an example in 1939 [8]. The example consists of the forms

$$f = x^2 + y^2 + 16z^2, \qquad g = 2x^2 + 2y^2 + 5z^2 - 2xz - 2yz,$$

which are easily seen to represent equally often every positive integer not an odd square. If s is odd, a result of Jacobi's [9] shows that

$$f(s^2) - g(s^2) = (-1)^{\frac{1}{2}(s-1)} 4s,$$

where $f(n)$ denotes the number of representations of n by f. Also,

$$f(s^2) + g(s^2) = 4\Pi\psi(p, a),$$

where

$$\psi(p, a) = (p^{a+1} - 1)/(p - 1) - (-1|p)(p^a - 1)/(p - 1),$$

and the product ranges over the prime-power decomposition $s = \Pi p^a$. From this it is easily seen that $g(s^2) = 0$ if every prime p in s satisfies $p \equiv 1 \bmod 4$. This example is especially interesting in that f and g are two forms in the same genus, and we are able to give exact simple formulas for the number of representations of any number in either form.

However, the theorem is true when $m = 3$ for a large variety of forms [10]; and, if we may make a conjecture, it is probably true for squarefree numbers n and every ternary f.

2. The asymptotic formula for the number of representations of n in f.

THEOREM 3. *Let $m \geqq 4$. For any positive integral m-ary form $f = \Sigma a_{ij}x_ix_j$ where a_{ii} and $2a_{ij}$ $(i, j = 1, \cdots, m)$ are integers, the number of representations $f(n)$ of n by f is given by the formula*

$$\frac{\pi^{\frac{1}{2}m} n^{\frac{1}{2}m-1} S(n)}{\Gamma(\frac{1}{2}m)\Delta^{\frac{1}{2}}} + O(n^{\frac{1}{2}m-1-1/18+\epsilon}), \tag{1}$$

where Δ denotes the determinant $|a_{ij}|$ of f,

$$S(n) = \prod_{p=2,3,5,7,\cdots} \chi(p), \qquad \chi(p) = \chi(p, f, n) = \lim_{r\to\infty} p^{-(m-1)r} f(n, p^r), \tag{2}$$

and $f(n, p^r)$ denotes the number of solutions $x_1, \cdots, x_m$ mod p^r of $f \equiv n \bmod p^r$.

Proof. We can write $f = b_1x_1^2 + \cdots + b_{s-1}x_{s-1}^2 + d_1\phi(x_s, \cdots, x_m)$, where ϕ is primitive. Let Δ_1 denote the determinant of ϕ; then $\Delta = b_1 \cdots b_{s-1}d_1^{m-s+1}\Delta_1$. Suppose, then, that (1) is known to be true if $s_1 \leqq s \leqq m$. Kloosterman's case is of course $s_1 = m$. Now let $s = s_1 - 1$. Let q denote an odd prime represented by ϕ and not dividing Δ. We can write $\phi = \sum_{i,j=s}^{m} a_{ij}x_ix_j$, $a_{ss} = q$.

Case I: the a_{sj} all integers. By completing squares we have the identity

$$qf = qb_1x_1^2 + \cdots + qb_{s-1}x_{s-1}^2 + d_1(qx_s + \sum_{j=s+1}^{m} a_{sj}x_j)^2 + d_1\psi, \tag{3}$$

where $\psi(x_{s+1}, \cdots, x_m) = \sum_{i,j=s+1}^{m} (qa_{ij} - a_{is}a_{sj})x_ix_j$. We introduce two forms:

$$f_1 = qb_1y_1^2 + \cdots + qb_{s-1}y_{s-1}^2 + d_1y_s^2 + d_1\psi(y_{s+1}, \cdots, y_m), \tag{4}$$

and the form f_2 got from f_1 by changing the coefficient of y_s^2 to q^2d_1. Now for every representation of qn in f_1 we have

$$y_s^2 \equiv -\psi(y_{s+1}, \cdots, y_m) \equiv \sum_{i,j=s+1}^{m} a_{is}y_i \cdot a_{sj}y_j \equiv (\sum_{j=s+1}^{m} a_{sj}y_j)^2 \pmod q.$$

Hence: $y_s \equiv \Sigma a_{sj}y_j \bmod q$ holds only by choice of sign of y_s if $q \nmid y_s$, and always if $q \mid y_s$. Accordingly,

$$\begin{aligned} f(n) &= N(n = f) = N(qn = qf) = N(qn = f_1;\ y_s \equiv \Sigma a_{sj}y_j \bmod q) \\ &= \tfrac{1}{2}\{N(qn = f_1) - N(qn = f_2)\} + N(qn = f_2), \text{ that is,} \end{aligned}$$

$$f(n) = \tfrac{1}{2}f_1(qn) + \tfrac{1}{2}f_2(qn). \tag{5}$$

If $p \neq q$, f_1 and f_2 are derived from qf by transformations of determinants prime to p, whence $f(n, p^r) = f_1(qn, p^r) = f_2(qn, p^r)$. Hence

$$\chi(p, f, n) = \chi(p, f_1, qn) = \chi(p, f_2, qn), \text{ if } p \neq q. \tag{6}$$

If $p = q$, we have (counting solutions of the congruences with care):

$$\begin{aligned} f(n, q^r) &= q^{-m} f(qn, q^{r+1}) = q^{-m+1} N(qn \equiv f_1 \bmod q^{r+1};\ y_s \equiv \Sigma a_{sj} y_j \bmod q) \\ &= q^{-m+1} \tfrac{1}{2}\{N(qn \equiv f_1 \bmod q^{r+1}) - q^{-1} N(qn \equiv f_2 \bmod q^{r+1})\} \\ &\quad + q^{-m+1} q^{-1} N(qn \equiv f_2 \bmod q^{r+1}), \end{aligned}$$

whence

$$f(n, q^r) = q^{-m+1}\{\tfrac{1}{2} f_1(qn, q^{r+1}) + \tfrac{1}{2} q^{-1} f_2(qn, q^{r+1})\}, \tag{7}$$

$$\chi(q, f, n) = \tfrac{1}{2}\chi(q, f_1, qn) + \tfrac{1}{2} q^{-1}\chi(q, f_2, qn). \tag{8}$$

Hence as (1) holds for f_1 and f_2, we have by (5),

$$f(n) = \{\Gamma(\tfrac{1}{2}m)\}^{-1} \pi^{\frac{1}{2}m} \Delta^{-\frac{1}{2}} n^{\frac{1}{2}m-1} \prod_{p \neq q} \chi(p) \cdot \lambda + O((qn)^{\frac{1}{2}m-1-1/18+\epsilon}),$$

where

$$\lambda = \tfrac{1}{2}(q^{m-2})^{-\frac{1}{2}} q^{\frac{1}{2}m-1}\chi(q, f_1, qn) + \tfrac{1}{2}(q^m)^{-\frac{1}{2}} q^{\frac{1}{2}m-1}\chi(q, f_2, qn) = \chi(q, f, n),$$

by (8). The induction is thus complete for Case I.

Case II: some coefficient $2a_{sj}$ is odd. We now use the identity

$$4qf = 4qb_1 x_1^2 + \cdots + 4qb_{s-1} x_{s-1}^2 + d_1(2qx_s + \Sigma 2a_{sj} x_j)^2 + d_1\psi,$$

where $\psi = \Sigma(4qa_{ij} - 4a_{is}a_{sj}) x_i x_j$. Besides the two forms

$$f_k = 4qb_1 y_1^2 + \cdots + 4qb_{s-1} y_{s-1}^2 + q^{2k-2} d_1 y_s^2 + d_1\psi(y_{s+1}, \cdots, y_m), \quad k = 1, 2, \tag{9}$$

we shall require the form f'_k obtained from f_k by changing y_s to $2y_s$, the form f_k'' obtained from f_k by expressing the condition that $\Sigma 2a_{sj} y_j$ is even (e. g. if $2a_{sm}$ is odd by replacing y_m in ψ by $2a_{s,s+1} y_{s+1} + \cdots + 2a_{s,m-1} y_{m-1} + 2y_m$), and the form f_k''' obtained by both the preceding operations.

If $4qn$ is represented in f_1, we now have only $y_s^2 \equiv (\Sigma 2a_{sj} y_j)^2 \bmod q$. Hence for any integer n,

$$\begin{aligned} f(n) &= N(4qn = 4qf) = N(4qn = f_1;\ y_s \equiv \Sigma 2a_{sj} y_j \bmod 2q) \\ &= \tfrac{1}{2} N(4qn = f_1, y_s \equiv \Sigma 2a_{sj} y_j \bmod 2) + \tfrac{1}{2} N(4qn = f_2, y_s \equiv \Sigma 2a_{sj} y_j \bmod 2). \end{aligned}$$

Hence it is easily seen that

$$(10) \qquad f(n) = \sum_{k=1}^{2} \{\tfrac{1}{2}f_k(4qn) - \tfrac{1}{2}f_k'(4qn) - \tfrac{1}{2}f_k''(4qn) + f_k(4qn)\}.$$

If $p \neq q$ and $p \neq 2$, we obviously have

$$f(n, p^r) = f_1(4qn, p^r) = f_1'(4qn, p^r) = \cdots = f_2'''(4qn, p^r),$$

whence $\chi(p)$ is the same in all cases. Evidently also $\chi(2)$ is the same for f_1 as for f_2, for f_1' as for f_2', for f_1'' as for f_2'', for f_1''' as for f_2'''; and $\chi(q)$ is the same for all four forms f_1, f_1', f_1'', f_1''', and again for the forms f_2, f_2', f''_2, f_2'''. Next,

$$f(n, q^r) = q^{-m}f(4qn, q^{r+1}) = q^{1-m}N(f_1 \equiv 4qn \bmod q^{r+1};\ y_s \equiv \Sigma 2a_{sj}y_j \bmod q)$$
$$= q^{1-m}\{\tfrac{1}{2}f_1(4qn, q^{r+1}) + \tfrac{1}{2}q^{-1}f_2(4qn, q^{r+1})\};$$

$$(11) \qquad \chi(q, f, n) = \tfrac{1}{2}\chi(q, f_1, 4qn) + \tfrac{1}{2}q^{-1}\chi(q, f_2, 4qn).$$

Also, $f(n, 2^r) = 4^{-m}N(4qf \equiv 4qn \bmod 2^{r+2}) = 2^{1-2m}N(f_1 \equiv 4qn \bmod 2^{r+2};$ $y_s \equiv \Sigma 2a_{sj}y_j \bmod 2) = 2^{1-2m}f_1(4qn, 2^{r+2}) - N(f_1 \equiv 4qn,\ y_s \text{ even})$ $- N(f_1 \equiv 4qn,\ \Sigma 2a_{sj}y_j \text{ even}) + 2N(f_1 \equiv 4qn,\ y_s \text{ and } \Sigma 2a_{sj}y_j \text{ even})$ whence

$$(12) \qquad \chi(2, f, n) = \tfrac{1}{2}\chi(2, f_1, 4qn) - \tfrac{1}{4}\chi(2, f_1', 4qn) - \tfrac{1}{4}\chi(2, f_1'', 4qn) + \tfrac{1}{4}\chi(2, f_1''', 4qn).$$

Substituting in (10) we obtain the required formula for $f(n)$ in this case also.

3. Proof of Theorem 1. This proceeds exactly as in *P* **3**. We have only to convert f into a convenient form-residue mod p^r; cf. Lemma 2 [11]. Corresponding to $P(12)$, with Δ in place of $abcd$, we can use known formulas [12] for $f(n, p^r)$ in the cases where $p \nmid 2\Delta$ to prove that $\chi(p) \geqq 1 - p^{-\frac{1}{2}m}$ if $p \nmid n$ and m is even, $\chi(p) \geqq (1 - p^{-\frac{1}{2}m})(1 - p^{-1})$ if $p \mid n$ and m is even, and $\chi(p) \geqq 1 - p^{1-m}$ if m is odd. Hence the product of the $\chi(p)$ for all odd primes not dividing Δ is easily seen to exceed $K/\log\log n$. The additional discussion for cases (14)-(17) will be found in *P*, **3**.

LEMMA 2. *Every integral quaternary form f is equivalent* mod p^r, *by a transformation of determinant prime to p, to a form of the type*

$$(13) \qquad p^{\alpha_1}a_1x_1^2 + \cdots + p^{\alpha_4}a_4x_4^2, \quad \bmod p^r,$$

$$0 \leqq \alpha_1 \leqq \alpha_2 \leqq \alpha_3 \leqq \alpha_4, \quad a_1a_2a_3a_4 \text{ prime to } p,$$

if $p > 2$. If $p = 2$, f is equivalent mod 2^r *either to* (13), *or to a form with one of the following residues* mod 2^r:

(14) $2^{\alpha_1}a_1x_1^2 + 2^{\alpha_2}a_2x_2^2 + 2^{\alpha_3}(2jx_3^2 + 2x_3x_4 + 2jx_4^2)$,

(15) $2^{\alpha_1}a_1x_1^2 + 2^{\alpha_2}(2jx_2^2 + 2x_2x_3 + 2jx_3^2) + 2^{\alpha_4}a_4x_4^2$,

(16) $2^{\alpha_1}(jx_1^2 + x_1x_2 + jx_2^2) + 2^{\alpha_3}a_3x_3^2 + 2^{\alpha_4}a_4x_4^2$,

(17) $2^{\alpha_1}(jx_1^2 + x_1x_2 + jx_2^2) + 2^{\alpha_3}(kx_3^2 + x_3x_4 + kx_4^2)$.

Here all a_i are odd, $j, k = 0$ or 1. Also, in (14), $0 \leqq \alpha_1 \leqq \alpha_2 < \alpha_3$; *in* (15), $0 \leqq \alpha_1 < \alpha_2 < \alpha_4$; *in* (16), $0 \leqq \alpha_1 \leqq \alpha_3 \leqq \alpha_4$; *and in* (17), $0 \leqq \alpha_1 \leqq \alpha_3$.

4. Proof of Theorem 2. We first extend Lemmas *P*1 and *P*2 as follows:

LEMMA 3. *The conditions that f be universal for p are as stated in Lemma P*1 *if f has the residue* (13). *If $p = 2$ and f has a residue* (14) *or* (15), *f is not universal for p. If $p = 2$ and f has a residue* (16) *or* (17), *f is universal for p if and only if, respectively:*

(18) $\alpha_1 = 0$, *and either* $j = 0$ *or* $\alpha_3 = 0$ *or* $\alpha_3 = 1 \geqq \alpha_4 - 2$;

(19) $\alpha_1 = 0$, *and either* $j = 0$ *or* $\alpha_3 = 0$ *or* $\alpha_3 = 1$.

LEMMA 4. *If f is universal for p, and f has a residue* (13), *the condition that f fail to represent zero p-adically is given in Lemma P*2. *If* (16) *and* (18), *or* (17) *and* (19), *hold, then f fails to represent zero p-adically if and only if, respectively:*

(20) $\alpha_1 = 0$, $j = 1$, $\alpha_3 = 1$, $\alpha_4 = 1$ *or* 3, *and* $a_3a_4 \equiv 3 \bmod 8$;

(21) $\alpha_1 = 0$, $j = 1$, $\alpha_3 = 1$, $k = 1$.

Again, Lemma *P*3 extends to the cases coming under (13). Also,

LEMMA 5. $2^s \mid f$ *implies* $2 \mid$ *all* x_i, *if* $s = 3 + \alpha_4$ *in* (20), *if* $s = 2$ *in* (21).

If $p > 2$, or if $p = 2$ and the cases corresponding to (21) hold, then if f does not represent one number n, f will not represent $p^{2k}n$, by Lemmas 5 and *P*3. Hence if f represents all large integers, $\det f \geqq R_1$.

Let $p = 2$ and f be given by (13). (a) If *P*(18) holds and f represents all large integers it must represent every $2n$, by Lemma *P*3. Now if $f = 2n$, Σx_i is even. Hence the transformation $x_1 = y_1$, $x_2 = y_2$, $x_3 = y_3$, $x_4 = y_1 + y_2 + y_3 + 2y_4$ replaces f by $2g$, where g is an integral form and g represents every n; by Lemma 1, $\det g \geqq R_1$, $\det f \geqq 4R_1$. (b) If *P*(20) holds, f is

obtainable (by changing x_4 to $2x_4$) from a form under case (a). Hence $\det f \leqq 16R_1$. (c) If $P(19)$ holds, and f represents all large integers, f represents every $4n$. In subcases (i) and (ii), $f = 4n$ implies $x_1 = y_1 + y_2$, $x_2 = y_1 - y_2$, $x_3 = y_3$, $x_4 = y_1 + y_2 + y_3 + 2y_4$ where the y_i are integers; and f becomes $4g$ where g represents all n; hence $\det f \leqq 16R_1$. In subcase (iii), $x_1 = y_1 + y_2$, $x_2 = y_1 - y_2$, $x_3 = y_3 + y_4$, $x_4 = y_3 - y_4$ yields the same result. (d) As in case (b), forms coming under $P(21)$-(23) lead to $\det f \leqq 64R_1$, $64R_1$, $256R_1$ respectively.

Next let $p = 2$ and f correspond to (20) with $\alpha_4 = 1$. Then f must represent every $4n$. If $f = 4n$, $x_1 = 2y_1$, $x_2 = 2y_2$, $x_3 = y_3 + y_4$, $x_4 = y_3 - y_4$, and we get $\det f \leqq 4R_1$. Finally, if $\alpha_4 = 3$ instead, $\det f \leqq 16R_1$.

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GENERAL THEORY OF SCALE CURVES.*

By EDWARD KASNER and JOHN DE CICCO.

1. Cartograms and scale curves. Let (x, y) denote general curvilinear coordinates of a point on a surface Σ and cartesian coordinates on a plane Π. This defines a point transformation T between the points of Σ and Π such that points correspond if they are represented by the same coordinates. We shall term any particular mapping of a surface Σ upon a plane Π a *cartogram*. Thus a cartogram depends not only on the surface Σ but also on the point transformation T. It is said to be conformal or non-conformal according as T is conformal or non-conformal.

The *scale function* $\sigma = ds/dS$ is the ratio of the differentials of arc lengths of the corresponding curves on Π and Σ respectively, under the transformation T. In general, σ depends not only on the point (x, y) but also on the slope $y' = dy/dx$; that is, $\sigma = \sigma(x, y, y')$. It is independent of the direction if, and only if, T is conformal. The scale σ is a mere constant only in the degenerate situation where Σ is developable and the mapping T is an unrolling of Σ upon Π followed by a similitude in Π.

A *scale curve* is the locus of a point on Σ or Π along which the scale σ does not vary. Thus the totality of scale curves for any given cartogram, is defined by $\sigma = \text{const.}$ For a non-conformal cartogram, there are ∞^2 scale curves. For a conformal cartogram, there are ∞^1 scale curves, except in the degenerate situation mentioned above where every curve is a scale curve.

Since we have already studied the simple-infinity of scale curves in conformal maps, we shall assume in the present paper that the mapping is not conformal. Thus the slope y' is explicitly present in the scale function σ. We shall study the geometry of systems of ∞^2 scale curves in the plane Π for non-conformal cartograms.

2. General summary. Upon solving the differential equation of the second order defining the ∞^2 scale curves for a given non-conformal cartogram for the second derivative $y'' = d^2y/dx^2$, it is found that y'' is expressed as a rational function of y' of the fifth degree with coefficients depending on six

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functions of (x, y). These six functions are not arbitrary but satisfy a system of two partial differential equations of the second order. (Theorem 3).

In general, there are essentially ∞^2 surfaces which possess the same family of ∞^2 curves as scale curves.

At any point of the plane Π, *there are three inflectional directions and two cuspidal directions for the system of* ∞^2 *scale curves.* (Theorem 1). The cuspidal directions are always orthogonal and coincide with the characteristic directions.

Through a point of the plane Π there pass ∞^1 scale curves. Upon constructing the osculating circles to these scale curves at the point, *the centers of curvature describe a general cubic curve (the central locus) which has a node at the fixed point, the tangent directions of the node coinciding with the characteristic directions.* (Theorem 2). Of course, this property is not completely characteristic for systems of scale curves.

A system of ∞^2 *scale curves is of the cubic (Lie-Liouville) type if, and only if, it is a velocity system.* (Theorem 5). In that event, the cubical locus degenerates into a straight line (the other two straight lines becoming the tangent lines of the characteristic directions). *If the scale curves form a natural family, then the characteristic curves form an isothermal net.* (Theorems 7 and 8).

We find *a new class of surfaces* Σ *for which there exists a map of* Σ *upon* Π *such that the scale curves coincide with the totality of* ∞^2 *straight lines of* Π. Corresponding to a given scale σ, there are ∞^1 straight scales all tangent to a conic section. As σ varies, the resulting ∞^1 conics form a confocal family.

There are essentially ∞^6 cartograms with straight lines as scales. The only such surfaces of constant gaussian curvature are developable surfaces. *Thus there exists no non-conformal map T of a sphere (or pseudo-sphere)* Σ *upon a plane* Π *such that the scale curves are all straight lines.* (Theorem 9). As a consequence of this, it follows that the only map, conformal or not, of a sphere upon a plane such that the scale curves are pictured by straight lines, is the Mercator projection, in which case there is only a simple-infinitude of straight scales.

3. The differential equation of the ∞^2 scale curves of a non-conformal cartogram. Let (x, y) denote general curvilinear coordinates of a point on a surface Σ and cartesian coordinates on a plane Π. The square of the linear element dS of Σ is

$$dS^2 = E(x, y)dx^2 + 2F(x, y)dxdy + G(x, y)dy^2, \tag{1}$$

where $H^2 = EG - F^2 > 0$, and the square of the linear element ds of Π is

$ds^2 = dx^2 + dy^2$. This defines a point transformation T between the points of Σ and Π such that points correspond if they are represented by the same coordinates. Since we are working with non-conformal cartograms, it follows that the conditions $F = 0$ and $E = G$, can never hold simultaneously.

The gaussian curvature G of the surface Σ is given by

$$(2) \quad \begin{aligned} 4H^4G = {} & E(G_x^2 - 2F_xG_y + E_yG_y) + G(E_y^2 - 2E_xF_y + E_xG_x) \\ & + F(E_xG_y - E_yG_x - 2F_xG_x + 4F_xF_y - 2E_yF_y) \\ & - 2H^2(E_{yy} - 2F_{xy} + G_{xx}). \end{aligned}$$

This will be found useful for our later work.

The scale function $\sigma = ds/dS$ is the ratio of the differentials of arc lengths of the corresponding curves on Π and Σ, respectively, under the transformation T. It is defined by the formula

$$(3) \quad \sigma^2 = \left(\frac{ds}{dS}\right)^2 = \frac{1 + y'^2}{E + 2Fy' + Gy'^2}.$$

Thus σ is a function of the lineal-element (x, y, y'). For a non-conformal cartogram, the slope y' is explicitly present in σ since the conditions $F = 0$ and $E = G$ do not hold simultaneously.

The scale curves are defined by $\sigma = \text{const.}$ *There are always* ∞^2 *scale curves in a non-conformal cartogram.* Upon eliminating the constant σ from the equation (3) by differentiation, we discover that the differential equation of the ∞^2 scale curves is

$$(4) \quad y'' = \frac{(1 + y'^2)[E_x + y'(E_y + 2F_x) + y'^2(2F_y + G_x) + y'^3G_y]}{2[-F + y'(E - G) + y'^2F]}.$$

Not every system of ∞^2 curves in the plane Π can represent the scales of a non-conformal cartogram, since (4) is of special algebraic form in the first derivative.

THEOREM 1. *At a point, there are three inflectional directions and two cuspidal directions for the scale curves. The cuspidal directions are always orthogonal and coincide with the characteristic directions.*

This may be deduced from the differential equation (4) of the ∞^2 scale curves. By a theorem of Tissot, the characteristic directions are those of the unique orthogonal net defined by

$$(5) \quad Fy'^2 + (E - G)y' - F = 0,$$

on Π which by the non-conformal transformation T is converted into an orthogonal net on Σ. This is termed the characteristic net.

4. The terminal ellipse. If, on the plane Π, the scale function σ is laid off along the corresponding direction y' at a fixed point (x, y), the end point will describe a locus, which is an ellipse. Let (X, Y) represent the cartesian coordinates of a point on the ellipse relative to the fixed point (x, y), which is, of course, the center of the ellipse. Then $\sigma^2 = X^2 + Y^2$ and $y' = Y/X$. By (3), it is found that the equation of this terminal ellipse is

$$EX^2 + 2FXY + GY^2 = 1. \tag{6}$$

The directions of the major and minor axes of this terminal ellipse are given by (5) so that they coincide with the characteristic directions. The semi-major and semi-minor diameters of the terminal ellipse are solutions of the equation

$$H^2\sigma^4 - (E + G)\sigma^2 + 1 = 0. \tag{7}$$

For a fixed σ, the preceding equation defines a curve such that at all points of this curve the corresponding terminal ellipses are congruent. This set of $2\infty^1$ curves is termed the *singular family* of curves.

It is noted that this singular family of $2\infty^1$ curves can degenerate into only ∞^1 curves if and only if, either, the mapping is conformal, or else the functions $E + G$ and $H^2 = EG - F^2$, are functionally related. For example, under any area preserving map the singular family is $E + G = \text{const.}$

5. Characterization of the system of ∞^2 scale curves. Let (X, Y) represent the cartesian coordinates relative to the point (x, y) of the center of curvature constructed at the point (x, y) of a scale curve. Then $X = -y'(1 + y'^2)/y''$, $Y = (1 + y'^2)/y''$. From these we find $y' = -X/Y$, $y'' = (X^2 + Y^2)/Y^3$. Substituting these values of y' and y'' into the differential equation (4) and simplifying, we obtain the equation

$$\begin{aligned} G_y X^3 - (2F_y + G_x)X^2Y + (E_y + 2F_x)XY^2 - E_x Y^3 \\ + 2[-FX^2 + (E - G)XY + FY^2] = 0. \end{aligned} \tag{8}$$

THEOREM 2. *The locus of the centers of curvature (the central locus) is a general cubic curve which has a node at the fixed point (x, y), the directions of the tangent lines at the node coinciding with the characteristic directions.*

This property is possessed not only by any family of ∞^2 scale curves but also by a more general class of families of curves. If a system of curves possesses the preceding property, then it must be defined by a differential equation of the form

$$y'' = \frac{(1 + y'^2)(\alpha + \beta y' + \gamma y'^2 + \delta y'^3)}{2(-\eta + \epsilon y' + \eta y'^2)}, \tag{9}$$

where $(\alpha, \beta, \gamma, \delta, \epsilon, \eta)$ are six functions of (x, y).

THEOREM 3. *The differential equation* (9) *will represent the* ∞^2 *scale curves of a non-conformal map of a surface* Σ *upon a plane* Π *if, and only if the six functions* $(\alpha, \beta, \gamma, \delta, \epsilon, \eta)$ *of* (x, y) *satisfy the two partial differential equations of the second order*

$$(10) \qquad \begin{aligned} &M_y = N_x, \\ &\alpha_y + N\alpha = \delta_x + \epsilon_{xy} + N\epsilon_x + \epsilon N_x + M(\delta + \epsilon_y + \epsilon N), \end{aligned}$$

where M *and* N *are defined by*

$$(11) \qquad \begin{aligned} (\epsilon^2 + 4\eta^2)M &= \epsilon(\alpha - \gamma) + 2\eta(\beta - \delta) - 4\eta\eta_x + 2\epsilon\eta_y - \epsilon\epsilon_x - 2\eta\epsilon_y, \\ (\epsilon^2 + 4\eta^2)N &= -2\eta(\alpha - \gamma) + \epsilon(\beta - \delta) - 2\epsilon\eta_x - 4\eta\eta_y + 2\eta\epsilon_x - \epsilon\epsilon_y. \end{aligned}$$

To prove this proposition, we proceed as follows. Upon comparing (9) with the differential equation (4) of the ∞^2 scale curves of a non-conformal cartogram, it follows that there exists a function $\rho(x, y) \neq 0$, such that

$$(12) \qquad \begin{gathered} F = \rho\eta, \quad E - G = \rho\epsilon, \quad E_x = \rho\alpha, \quad E_y + 2F_x = \rho\beta, \\ 2F_y + G_x = \rho\gamma, \quad G_y = \rho\delta. \end{gathered}$$

From these equations we eliminate F and G and obtain

$$(13) \qquad \begin{aligned} E_x &= \rho\alpha = \rho(\gamma - 2\eta_y + \epsilon_x) + \epsilon\rho_x - 2\eta\rho_y, \\ E_y &= \rho(\beta - 2\eta_x) - 2\eta\rho_x = \rho(\delta + \epsilon_y) + \epsilon\rho_y. \end{aligned}$$

These equations can be solved for ρ_x/ρ and ρ_y/ρ, giving the solutions

$$(14) \qquad \rho_x/\rho = M, \qquad \rho_y/\rho = N,$$

where M and N are defined by (11).

Substituting these into (13), we find

$$(15) \qquad \begin{aligned} E_x &= \rho\alpha = \rho(\gamma - 2\eta_y + \epsilon_x + \epsilon M - 2\eta N), \\ E_y &= \rho(\beta - 2\eta_x - 2\eta M) = \rho(\delta + \epsilon_y + \epsilon N). \end{aligned}$$

Upon imposing the compatibility conditions on equations (14) for ρ and on equations (15) for E, we obtain the conditions (10). This completes the proof of our Theorem 3.

COROLLARY. *There are essentially* ∞^2 *surfaces which possess the same family of* ∞^2 *scale curves.*

A consideration of equations (12), (14), and (15) will show that if the surface Σ with the linear element (1) is mapped upon the plane Π such that

the differential equation (4) represents the scale curves, then the linear element dS' of any other surface Σ' with the same scale curves must be given by

$$dS'^2 = adS^2 + bds^2, \tag{16}$$

where $a \neq 0$ and b are constants. This proves the corollary stated above.

THEOREM 4. *The only point transformations which convert the class of systems of scale curves (or the class (9)) into itself, are those of the conformal group.*

This result follows from the fact that if the classes (4) or (9) are preserved, then certainly the minimal lines are preserved. That systems of scale curves are converted under conformalities into systems of scale curves, follows from a consideration of the scale function σ, given by (3).

It is remarked that the results of this section are valid only for the general case where the fraction of (4) is not reducible to lower terms.

6. Scale curves of the velocity type. Before continuing further, we shall consider briefly the second order differential equations of the cubic type and also of the velocity type.

A differential equation of the cubic type is of the form $y'' = \alpha + \beta y' + \gamma y'^2 + \delta y'^3$, where $(\alpha, \beta, \gamma, \delta)$ are functions of (x, y) only. This has been studied extensively by Lie, R. Liouville, Tresse, Kasner, and Wilczynski. With respect to the group of arbitrary point transformations, this type is called a differential equation of the first rank.

A general velocity system is defined by the differential equation $y'' = (1 + y'^2)(\psi - y'\phi)$, where ϕ and ψ are functions of (x, y). This has the property that the central locus is always a straight line.

THEOREM 5. *A system of ∞^2 scale curves can be of the cubic (Lie-Liouville) type if and only if it is a velocity system. The necessary and sufficient conditions for this are*

$$\begin{aligned} G_y(E - G) &= F(E_x + G_x + 2F_y), \\ -E_x(E - G) &= F(E_y + G_y + 2F_x). \end{aligned} \tag{17}$$

From (4), it is evident that $(1 + y'^2)$ and the denominator can not have a common root in y', since the roots of the denominator are both real. Hence for (4) to be of the cubic type, the denominator must be a factor of the cubic factor of the numerator and this demonstrates that it must be a velocity system.

Hence (4) must be of the form $y'' = (1 + y'^2)(\psi - y'\phi)/2$, and we find

$$(18)\qquad \begin{aligned} \psi F &= G_y, & \psi(E-G) &= E_x + G_x + 2F_y, \\ \phi F &= E_x, & -\phi(E-G) &= E_y + G_y + 2F_x. \end{aligned}$$

The elimination of ϕ and ψ from these equations leads to the conditions (17).

Note that if the parametric curves of the surface Σ are orthogonal, that is, $F = 0$, and if it be demanded that y'' be a polynomial in y', it does not follow necessarily that the scale curves must be of the cubic type and hence a velocity system. In this case, if we impose the condition that it be of the cubic type, then $G_y = E_x = 0$, and hence it is a velocity system. However, if the parametric curves of the surface Σ are not orthogonal, that is, $F \neq 0$, and if it be required that y'' be a polynomial in y', it does follow that the ∞^2 scale curves are of the cubic type.

Under the conditions of Theorem 5, if $F \neq 0$, the differential equation of the ∞^2 scale curves is

$$(19)\qquad y'' = (1/2F)(1 + y'^2)(-E_x + y'G_y);$$

and if $F = 0$, we find

$$(20)\qquad y'' = [1/2(E-G)](1 + y'^2)(E_y + y'G_x).$$

THEOREM 6. *The conditions* (17) *are the necessary and sufficient conditions for the cubical locus* (8) *to degenerate into three straight lines, two of which must be the tangent lines of the characteristic curves at the fixed point.*

This may be obtained by imposing the condition on (8) that it consist of three straight lines. If $F \neq 0$, the third straight line is

$$(21)\qquad G_yX + E_xY - 2F = 0;$$

and, if $F = 0$, the third straight line is

$$(22)\qquad -G_xX + E_yY + 2(E-G) = 0.$$

THEOREM 7. *If the scale curves form a natural family, then the characteristic curves are an isothermal net.*

To prove this result, we use (9) and Theorem 3. For (9) to represent the velocity system $y'' = (1 + y'^2)(\psi - y'\phi)/2$, we find

$$(23)\qquad \alpha = -\psi\eta, \quad \beta = \psi\epsilon + \phi\eta, \quad \gamma = \psi\eta - \epsilon\phi, \quad \delta = -\phi\eta.$$

Substituting these values in (11), we find that M and N are given by

$$(24)\qquad \begin{aligned} M &= \phi - \tfrac{1}{2}(\partial/\partial x)\log(\epsilon^2 + 4\eta^2) + (\partial/\partial y)\arctan(2\eta/\epsilon), \\ N &= \psi - \tfrac{1}{2}(\partial/\partial y)\log(\epsilon^2 + 4\eta^2) - (\partial/\partial x)\arctan(2\eta/\epsilon). \end{aligned}$$

Placing these values into the first of conditions (10), we have

$$(25) \qquad -\phi_y + \psi_x = (\partial^2/\partial x^2 + \partial^2/\partial y^2) \arctan (2\eta/\epsilon).$$

The condition for a natural family is $\phi_y = \psi_x$. From this equation, it follows, by a theorem of Lie concerning isothermal families, that the characteristic curves form an isothermal net.

THEOREM 8. *If the scale curves of a non-conformal cartogram form a velocity system and if the characteristic curves are an isothermal net, then the scale curves are a natural family.*

For if the characteristic curves are an isothermal family then the right-hand side of (25) is zero. Hence $\phi_y = \psi_x$ and the scale curves form a natural family.

7. Non-conformal cartograms with rectilinear scales. We shall state and prove the following result. We obtain a new class of surfaces.

THEOREM 9. *The class of surfaces* Σ *for which there exists a map upon the plane* Π *such that the scale curves coincide with the totality of* ∞^2 *straight lines is*

$$(26) \quad dS^2 = a_0(ydx - xdy)^2 + 2(a_1dx + b_1dy)(ydx - xdy) + a_2dx^2 + 2c_2dxdy + b_2dy^2.$$

Thus there are essentially ∞^6 *such non-conformal cartograms.*

For this class of surfaces, we find

$$(27) \quad H^2 = (a_0a_2 - a_1^2)x^2 + 2(a_0c_2 - a_1b_1)xy + (a_0b_2 - b_1^2)y^2 + 2(a_1c_2 - a_2b_1)x + 2(a_1b_2 - b_1c_2)y + (a_2b_2 - c_2^2).$$

By (2), the gaussian curvature G of this class of surfaces is

$$(28) \qquad H^4G = -2a_0H^2 + a_0(c_2^2 - a_2b_2) + a_2b_1^2 - 2a_1b_1c_2 + a_1^2b_2.$$

The proof of Theorem 9 is as follows. If the scale curves, represented by the differential equation (4), are all straight lines, we must have

$$(29) \qquad E_x = 0, \quad E_y + 2F_x = 0, \quad 2F_y + G_x = 0, \quad G_y = 0.$$

The condition of compatibility for F yields $-2F_{xy} = E_{yy} = G_{xx}$. These together with other conditions yield

$$(30) \qquad E = a_0y^2 + 2a_1y + a_2, \quad G = a_0x^2 - 2b_1x + b_2, \quad F = -a_0xy - a_1x + b_1y + c_2.$$

These then lead to the equation (26) and the proof is complete.

THEOREM 10. *The only surfaces* Σ *of constant gaussian curvature of the class* (26) *are the developable ones. Thus there exists no non-conformal map of a sphere or pseudo-sphere upon a plane such that the scale curves coincide with the totality of the* ∞^2 *straight lines.*

By (28), it follows that if G is identically constant, then H is identically constant. By (27), we must have

$$(31)\quad a_0a_2 - a_1^2 = a_0c_2 - a_1b_1 = a_0b_2 - b_1^2 = a_1c_2 - a_2b_1 = a_1b_2 - b_1c_2 = 0.$$

If $a_0 \neq 0$, these five equations will be satisfied by

$$(32)\qquad a_2 = a_1^2/a_0, \quad c_2 = a_1b_1/a_0, \quad b_2 = b_1^2/a_0.$$

However in this case, we shall have $H^2 = a_2b_2 - c_2^2 = 0$. This contradiction shows that $a_0 = 0$.

Since $a_0 = 0$, it follows by (31) that $a_0 = a_1 = b_1 = 0$. By (28), it is seen that G is identically zero. Hence the surface Σ is developable and Theorem 10 is proved.

It is remarked that for a developable surface Σ of Theorem 10, the map T is an unrolling of Σ upon the plane Π followed by an affine transformation in Π. To a given scale σ, corresponds a parallel pencil of straight lines.

Now let us consider the general class of surfaces (26) *for which* $a_0 \neq 0$.

By an appropriate similitude, the class (26) may be reduced to the form

$$(33)\qquad dS^2 = (ydx - xdy)^2 + adx^2 + bdy^2.$$

By (27) and (28), we find

$$(34)\qquad H^2 = ax^2 + by^2 + ab, \qquad H^4G = -2H^2 - ab.$$

It is observed that a and b cannot both be zero. If one of these is zero, then the other must be positive. The locus $H = 0$ can be an imaginary or real ellipse according as a and b are both positive or both negative, a hyperbola if a and b are of oppositive signs, and two coincident straight lines if a or b is zero.

The curves $G = \text{const.}$, are conic sections similar and similarly placed to $H = 0$.

The surface Σ exists for all values of (x, y) if $H = 0$ is an imaginary ellipse, for the interior points of the real ellipse $H = 0$, for the points of the side of the hyperbola $H = 0$ not containing the center, and for all the points on both sides of the line $H = 0$.

The scale σ satisfies the equation

$$(35) \qquad (y - xy')^2 + a + by'^2 = (1/\sigma^2)(1 + y'^2).$$

Hence to a given scale σ correspond ∞^1 straight lines. These straight lines will all be tangent to the conic

$$(36) \qquad x^2/(1/\sigma^2 - b) + y^2/(1/\sigma^2 - a) = 1,$$

confocal with the conic $H = 0$.

Finally let us consider the general class of surfaces (26) *for which* $a_0 = 0$.

By an appropriate similitude, the class (26) may be reduced to the form

$$(37) \qquad dS^2 = 2dy(ydx - xdy) + adx^2.$$

By (27) and (28), we find

$$(38) \qquad H^2 = -y'' - 2ax, \qquad H^4 G = a.$$

It is noted that $H = 0$ is a parabola. The surface Σ exists for all points in the interior of this parabola $H = 0$.

The curves $G = \text{const.}$ are parabolas congruent to the parabola $H = 0$ and having the same axis as that of $H = 0$.

The scale σ satisfies the equation

$$(39) \qquad 2y'(y - xy') + a = (1/\sigma^2)(1 + y'^2).$$

Hence to a given scale σ correspond ∞^1 straight lines. These straight lines will be all tangent to the parabola

$$(40) \qquad y^2 + (a - 1/\sigma^2)(2x + 1/\sigma^2) = 0,$$

confocal with the parabola $H = 0$.

THEOREM 11. *In the general class* (26), *where at least one of the constants* a_0, a_1, b_1, *is not zero, the surface* Σ *exists on one side of the conic section* $H = 0$. *The gaussian curvature* G *is constant along the conic sections similar and similarly placed to the ellipse or hyperbola* $H = 0$, *or along the parabolas congruent to and having the same axis as* $H = 0$, *or along the lines parallel to the line* $H = 0$. *To a given scale* σ *there correspond* ∞^1 *straight scales all tangent to a conic confocal with* $H = 0$.

This theorem summarizes all the work of the last few paragraphs.

Our new general theory which has been developed in this paper will be applied to the classic map projections or cartograms (area-preserving, azimuthal, etc.) in later papers. The family of ∞^2 scale curves is usually quite complicated.

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PROJECTIVE ALGEBRA I.*

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1. Projective algebra of subsets of a direct product. Let X and Y be any two sets of points and $[X; Y]$ the class of all pairs $[x; y]$, $x \in X$, $y \in Y$. Fix $[x_0; y_0]$ arbitrarily in $[X; Y]$. Suppose now that $\mathfrak{B}$ is a boolean algebra of subsets A of $[X; Y]$, including as the identity I the entire set $[X; Y]$, and the null set 0. Define for every $A \in \mathfrak{B}$, the set A_x as the class of all pairs $[x; y_0]$ for which there exists a $y \in Y$ such that $[x; y] \in A$. Define A_y similarly. For every $A = [X_1; y_0]$ in $\mathfrak{B}$, $X_1 \subset X$, and $B = [x_0; Y_1] \in \mathfrak{B}$, $Y_1 \subset Y$, define the "product" set $A \square B$ as the class of all pairs $[x_1; y_1]$, $x_1 \in X_1$, $y_1 \in Y_1$. Note that if A or B is 0, $A \square B = 0$. We say that $\mathfrak{B}$ is a *projective algebra of subsets of the product* $[X; Y]$ in case for all A of $\mathfrak{B}$, A_x and A_y also are in $\mathfrak{B}$, and $A \square B$ is in $\mathfrak{B}$ for A, B of the above type. Thus we demand that the boolean algebra $\mathfrak{B}$ be closed under projection and $\square$-product formation. In particular, the boolean algebra of *all* subsets of $[X; Y]$ is such a projective algebra.

It may be remarked that the following is a modest beginning for a study of logic with quantifiers from a boolean point of view, since, for example, the set A_x is essentially (the y_0 being a dummy) the class of all x for which there exists a y such that the proposition $A(x, y)$ is true.

Our object is to discover a set of properties of the above model which, adopted as postulates for an abstract "projective algebra," will permit a representation theorem to the effect that every such abstract algebra is isomorphic, with preservation of union, intersection, complement, x- and y-projection, and $\square$-product, to a projective algebra of subsets of some direct product $[X; Y]$. This is accomplished only for the atomic case in the present paper. The authors hope to study the general problem and related matters subsequently.

We have derived many of the more immediate properties of the abstractly defined projective algebra, and have shown, in particular, that every such algebra is embeddable in a complete ordered projective algebra.

The essential properties of the above model which we use later as axioms are the following.

PROPERTY 1. $(A \cup B)_x = A_x \cup B_x$; $(A \cup B)_y = A_y \cup B_y$.

* Received February 16, 1945.

For if $[x; y_0] \in (A \cup B)_x$, then for some y, $[x; y]$ is in A or B; hence $[x; y_0]$ is in A_x or B_x. Similarly, $A_x \cup B_x \subset (A \cup B)_x$. If either A or B is 0, Property 1 is trivial. (See Property 3.)

PROPERTY 2. *$I_{xy} = [x_0; y_0] = I_{yx}$, where $[x_0; y_0]$ is an atom in $\mathfrak{B}$, i. e., an element minimal over 0 in $\mathfrak{B}$.*

For $I_x = [X; y_0]$, and I_{xy} is the class of all $[x_0; y]$ such that for some x, $[x; y] \in I_x$; hence $I_{xy} = [x_0; y_0]$.

PROPERTY 3. *$A_x = 0$ if and only if $A = 0$, and similarly for y-projection.*

If $A = 0$, then the set of all $[x; y_0]$ for which there exists an $[x; y]$ in A is empty. If $A \neq 0$, there is at least one point $[x; y] \in A$, and, hence, at least one $[x; y_0]$ in A_x.

PROPERTY 4. $A_{xx} = A_x$; $A_{yy} = A_y$.

If $[x_1; y_0] \in A_x$, then $[x_1, y_0] \in A_{xx}$. If $[x_1; y_0] \in A_{xx}$ then for some y, $[x_1; y] \in A_x$; hence $y = y_0$, and $[x_1; y_0] \in A_x$. If $A = 0$, Property 4 is trivial.

PROPERTY 5. *For $0 \neq A = [X_1; y_0] \subset I_x$, $0 \neq B = [x_0; Y_1] \subset I_y$, the set $A \square B = \{[x_1; y_1]; x_1 \in X_1, y_1 \in Y_1\}$ has the property*

$$(A \square B)_x = A, \qquad (A \square B)_y = B,$$

and if $S \in \mathfrak{B}$, $S_x = A$, $S_y = B$, then $S \subset A \square B$.

For $(A \square B)_x$ is the set of all $[x; y_0]$ such that $[x; y] \in [X_1; Y_1]$ for some y; hence $(A \square B)_x = [X_1; y_0] = A$. Suppose now $S_x = [X_1; y_0]$, $S_y = [x_0; Y_1]$ and $[x; y] \in S$. Then $[x; y_0] \in S_x = [X_1; y_0]$ and $[x_0; y] \in S_y = [x_0; Y_1]$. Hence $x \in X_1$, $y \in Y_1$, and $[x; y] \in [X_1; Y_1] = A \square B$. Thus $S \subset A \square B$.

PROPERTY 6. *$I_x \square p_0 = I_x$, and $p_0 \square I_y = I_y$, where $p_0 = [x_0; y_0]$.*

This is immediate from the $\square$-definition.

PROPERTY 7. *For $A_1 = [X_1; y_0]$, $A_2 = [X_2; y_0]$ in $\mathfrak{B}$, one has $(A_1 \cup A_2) \square I_y = (A_1 \square I_y) \cup (A_2 \square I_y)$, and similarly for $B_i = [x_0; Y_i]$ and I_x.*

For $(A_1 \cup A_2) \square I_y = [X_1 \cup X_2; y_0] \square [x_0; Y] = [X_1 \cup X_2; Y]$ and $(A_1 \square I_y) \cup (A_2 \square I_y) = [X_1; Y] \cup [X_2; Y] = [X_1 \cup X_2; Y]$.

If A_1 or A_2 is 0, Property 7 is trivial.

2. Projective algebra. Let $\mathfrak{B}$ be a boolean algebra with unit i, zero 0, $i > 0$, so that for all $a \in \mathfrak{B}$, $0 \leqq a \leqq i$. $\mathfrak{B}$ is said to be a *projective algebra*

if two mappings $a \to a_x$ and $a \to a_y$ of $\mathfrak{B}$ into $\mathfrak{B}$ are defined, satisfying the following postulates.

*P*1. $(a \vee b)_x = a_x \vee b_x$; $(a \vee b)_y = a_y \vee b_y$.

*P*2. $i_{xy} = p_0 = i_{yx}$ *where* p_0 *is an atom of* $\mathfrak{B}$, *that is, an element minimal over* 0 *in* $\mathfrak{B}$.

*P*3. $a_x = 0$ *if and only if* $a = 0$, *and* $a_y = 0$ *if and only if* $a = 0$.

*P*4. $a_{xx} = a_x$; $a_{yy} = a_y$.

*P*5. *For* $0 < a \leqq i_x$, $0 < b \leqq i_y$, *there exists an element* $a \square b$ *such that* $(a \square b)_x = a$, $(a \square b)_y = b$, *with the property that* $t \in \mathfrak{B}$, $t_x = a$, $t_y = b$ *implies* $t \leqq a \square b$.

*P*6. $i_x \square p_0 = i_x$; $p_0 \square i_y = i_y$.

*P*7. $0 < a_1$, $a_2 \leqq i_x$ *implies* $(a_1 \vee a_2) \square i_y = (a_1 \square i_y) \vee (a_2 \square i_y)$; *and* $0 < b_1$, $b_2 \leqq i_y$ *implies* $i_x \square (b_1 \vee b_2) = (i_x \square b_1) \vee (i_x \square b_2)$.

We prove a number of immediate consequences of these postulates.

*C*1. $a \leqq b$ *implies* $a_x \leqq b_x$ and $a_y \leqq b_y$.

From $a \vee b = b$ one has (*P*1) $(a \vee b)_x = b_x = a_x \vee b_x$.

*C*2. *For all* $a \in \mathfrak{B}$, $a_x \leqq i_x$, $a_y \leqq i_y$.

By *C*1, $a \leqq i$; hence $a_x \leqq i_x$.

*C*3. $i_x \wedge (a_x)' \leqq (a')_x$ *and similarly for y-projection.*

For $i = a \vee a'$, $i_x = a_x \vee (a')_x$ (*P*1), $i_x \wedge (a_x)' = (a_x \vee (a')_x) \wedge (a_x)' = 0 \vee ((a')_x \wedge (a_x)') \leqq (a')_x$.

*C*4. $(a \wedge b)_x \leqq a_x \wedge b_x$; $(a \wedge b)_y \leqq a_y \wedge b_y$.

For $(a \wedge b) \leqq a, b$, hence (*C*1) $(a \wedge b)_x \leqq a_x, b_x$.

*C*5. $a > 0$ *implies* $a_{xy} = p_0 = a_{yx}$.

Since $a \leqq i$, $a_{xy} \leqq i_{xy} = p_0$ (*C*1, *P*2). Since p_0 is an atom, $a_{xy} = p_0$ for $a_{xy} = 0$ is impossible by *P*3.

*C*6. $(p_0)_x = p_0 = (p_0)_y \leqq i_x \wedge i_y$.

For $p_0 = i_{xy}$ and hence $(p_0)_y = i_{xyy} = i_{xy} = p_0$, (*P*2, *P*4). The final inequality follows from *C*2.

In connection with the $\square$-product, it is convenient to make two definitions.

*D*1. For $a \leqq i_x$, $b \leqq i_y$, define $a \square 0$, $0 \square b$, and $0 \square 0$ all to be 0.

It is seen that for all of these the x- and y-projections are 0.

*D*2. For all $a \in \mathfrak{B}$, define the "closure" a^* of a to be $a_x \square a_y$.

*C*7. $a \square b = 0$ *if and only if* $a = 0$ *or* $b = 0$.

For if $a > 0$, $b > 0$, by *P*5, $(a \square b)_x = a > 0$, hence $a \square b > 0$ by *P*3.

C8. The following are equivalent: (a) $a \leqq i_x$; (b) *there is a* $t \in \mathfrak{B}$ *such that* $t_x = a$; (c) $a_x = a$; *and similarly for y-projections.*

(a) implies (b), for if $a = 0$, $a_x = a$ by *P*3, whereas for $a > 0$, $(a \sqcap i_y)_x = a$ by *P*5. Next, (b) implies (c), since $t_x = a$ yields $t_{xx} = a_x = t_x = a$ (*P*4). Finally, (c) implies (a) by *C*2.

C9. $0 < a_1 \leqq a_2 \leqq i_x$, $0 < b_1 \leqq b_2 \leqq i_y$ *if and only if* $0 < a_1 \sqcap b_1 \leqq a_2 \sqcap b_2$.

If $0 < a_1 \sqcap b_1 \leqq a_2 \sqcap b_2$, then (*C*1, *P*5) $0 < a_1 \leqq a_2$, $0 < b_1 \leqq b_2$. Conversely, assume the first inequalities and let $u = (a_1 \sqcap b_1) \vee (a_2 \sqcap b_2)$. Then $u_x = a_1 \vee a_2 = a_2$, $u_y = b_1 \vee b_2 = b_2$ (*P*1, *P*5); hence $u \leqq a_2 \sqcap b_2$, by *P*5, and $a_1 \sqcap b_1 \leqq a_2 \sqcap b_2$.

C10. The (*) *operation has the following closure properties*:

$$0^* = 0; \quad a^* \geqq a; \quad a^{**} = a^*; \quad a \geqq b \text{ implies } a^* \geqq b^*; \quad i^* = i.$$

First, $0^* = 0_x \sqcap 0_y = 0 \sqcap 0 = 0$. Second, $a^* = a_x \sqcap a_y \geqq a$ by the maximality property of $\sqcap$-product in *P*5, for $a > 0$. For $a = 0$, this becomes trivial. Third, $a^{**} = (a_x \sqcap a_y)_x \sqcap (a_x \sqcap a_y)_y = a_x \sqcap a_y = a^*$, for $a \geqq 0$. Fourth, $a \leqq b$ implies $a^* \leqq b^*$ by *C*1, *C*9. Fifth, $i^* \geqq i$, hence $i^* = i$.

C11. $a_1, a_2 \leqq i_x$, $b_1, b_2 \leqq i_y$ *imply* $(a_1 \wedge a_2) \sqcap (b_1 \wedge b_2) = (a_1 \sqcap b_1) \wedge (a_2 \sqcap b_2)$.

If any one of the a_i, b_i is 0, *C*11 is trivial. Otherwise, $a_1 \wedge a_2 \leqq a_1, a_2$, $b_1 \wedge b_2 \leqq b_1, b_2$, hence $(a_1 \wedge a_2) \sqcap (b_1 \wedge b_2) \leqq a_1 \sqcap b_1$, $a_2 \sqcap b_2$ by *C*9. Hence the left member is contained in the right. Let t denote the right member. We have $t_x \leqq a_1 \wedge a_2$, $t_y \leqq b_1 \wedge b_2$ (*C*4, *P*5); hence $t \leqq t^* = t_x \sqcap t_y \leqq (a_1 \wedge a_2) \sqcap (b_1 \wedge b_2)$ by *C*10, *C*9.

C12. $a \leqq i_x$, $b \leqq i_y$ *implies* $a \sqcap b = (a \sqcap i_y) \wedge (i_x \sqcap b)$.

In *C*11, let $a_1 = a$, $a_2 = i_x$, $b_1 = i_y$, $b_2 = b$.

C13. $t \leqq a \sqcap b$ *implies* $t_x \leqq a$, $t_y \leqq b$.

This requires only *C*1 and *P*5 for $t > 0$. It is trivial for $t = 0$.

C14. $((a_1 \sqcap b_1) \vee (a_2 \sqcap b_2))^* = (a_1 \vee a_2) \sqcap (b_1 \vee b_2)$ *for* a_i, b_i *not* 0.

We require *P*1, *P*5.

C15. $a \leqq i_x$ *implies* $(a \sqcap i_y) \wedge i_x = a$.

For $a \leqq i_x$, $a \leqq a^* = a \sqcap p_0 < a \sqcap i_y$. Hence $a \leqq (a \sqcap i_y) \wedge i_x$. Now let d be this intersection. Then $d \leqq i_x$, $d_x = d \leqq a \wedge i_x = a$.

C16. All a, b *satisfying* $0 < a \leqq i_x$, $0 < b \leqq i_y$ *are closed, i. e.,* $a \sqcap p_0 = a$, $p_0 \sqcap b = b$.

$a^* = a \sqcap p_0$ since $a_x = a$ and $a_y = a_{xy} = p_0$. By *C*12, $a \sqcap p_0 = (a \sqcap i_y) \wedge (i_x \sqcap p_0) = (a \sqcap i_y) \wedge i_x$ by *P*6. Then $a \sqcap p_0 = a$ by *C*15.

C17. $(p_0)^* = p_0 = p_0 \sqcap p_0 = (p_0 \sqcap i_y) \wedge (i_x \sqcap p_0) = i_x \wedge i_y$.

This follows from $C6$, $C16$, $C12$, $P6$.

$C18$. *If* $t_y = p_0$ *then* $t \leqq i_x$. *If* $t_x = p_0$ *then* $t \leqq i_y$.

For $t \leqq t^* \leqq i_x \square p_0 = i_x$ by $C10$, $C9$, $P6$.

$C19$. For a_1, $a_2 \leqq i_x$, $b \leqq i_y$, *one has* $(a_1 \vee a_2) \square b = (a_1 \square b) \vee (a_2 \square b)$. *Similarly for* b_1, $b_2 \leqq i_y$, $a \leqq i_x$.

By $C12$, $P7$, $(a_1 \vee a_2) \square b = ((a_1 \vee a_2) \square i_y) \wedge (i_x \square b) = ((a_1 \square i_y) \vee (a_2 \square i_y)) \wedge (i_x \square b) = ((a_1 \square i_y) \wedge (i_x \square b)) \vee ((a_2 \square i_y) \wedge (i_x \square b)) = (a_1 \square b) \vee (a_2 \square b)$.

$C20$. *For* a_1, $a_2 \leqq i_x$, b_1, $b_2 \leqq i_y$ *one has* $(a_1 \vee a_2) \square (b_1 \vee b_2) = (a_1 \square b_1) \vee (a_1 \square b_2) \vee (a_2 \square b_1) \vee (a_2 \square b_2)$.

This requires two applications of $C19$.

$C21$. $(a \square i_y)' = (a' \wedge i_x) \square i_y$; $(i_x \square b)' = i_x \square (b' \wedge i_y)$.

For $(a \square i_y) \vee ((a' \wedge i_x) \square i_y) = (a \vee (a' \wedge i_x)) \square i_y = i_x \square i_y = i_y$ by $P7$. and $(a \square i_y) \wedge ((a' \wedge i_x) \square i_y) = (a \wedge (a' \wedge i_x)) \square i_y = 0 \square i_y = 0$, by $C11$. Since complementation is unique in a boolean algebra, $C21$ follows.

$C22$. $(a \square b)' = ((a' \wedge i_x) \square i_y) \vee (i_x \square (b' \wedge i_y))$.

This follows from $C12$ and $C21$.

$C23$. *For* $c = c^*$, $c_y < i_y$, *one has* $(c')_x = i_x$.

If $c = 0$, this is trivial. If $0 < c = c_x \square c_y$, then $c' = (((c_x)' \wedge i_x) \square i_y) \vee (i_x \square ((c_y)' \wedge i_y))$ by $C22$, and $(c')_x = (((c_x)' \wedge i_x) \square i_y)_x \vee i_x = i_x$ by $P1$, $P5$. We need to know that $(c_y)' \wedge i_y \neq 0$ but this is so since $c_y < i_y$.

$C24$. $a \leqq i_x$, $p_0 < i_y$ *implies* $(a')_x = i_x$.

For by $C16$, $a = a^*$, and $a_x = a$, $a_{xy} = p_0 = a_y$. $C24$ now follows from $C23$. (For $a = 0$, $C24$ is trivial).

$C25$. *If* $p_0 < i_y$ *then* $((i_x)')_x = i_x$, *and similarly for* i_y.

Let $a = i_x$ in $C24$.

$C26$. $p_0 = i_y$ *if and only if* $i = i_x$.

If $p_0 = i_y$, then $i = i_x \square i_y = i_x \square p_0 = i_x$ ($C10$, $P6$). If $i = i_x$, then $i_y = i_{xy} = p_0$.

$C27$. $((i_x)')_y = p'_0 \wedge i_y$.

If $p_0 = i_y$, by $C26$, $((i_x)')_y = ((i)')_y = 0_y = 0 = (i_y)' \wedge i_y = p'_0 \wedge i_y$. If $p_0 < i_y$, one has from $i_x = i_x \square p_0$ ($P6$) that $(i_x)' = (i_x \square p_0)' = i_x \square (p'_0 \wedge i_y)$ where $p'_0 \wedge i_y \neq 0$ ($C21$). Hence $((i_x)')_y = p'_0 \wedge i_y$ by $P5$.

$C28$. *The class of all* x*-projections is an ideal in* $\mathfrak{B}$ *and the correspondence* $a_x \to a_x \square i_y$ *is an isomorphism.*

The set of all a_x, $a \in \mathfrak{B}$ is exactly the set of all $a \leqq i_x$, by $C8$, and hence is an ideal, since the union of any two a_1, $a_2 \leqq i_x$ is $\leqq i_x$, as is $a_1 \wedge c$ for all

$c \in \mathfrak{B}$. Moreover the correspondence above preserves union ($P7$), intersection ($C11$), and complement ($C21$).

$C29$. *For* $c \in \mathfrak{B}$, $0 < a \leqq c_x$ *implies* $(a \square i_y) \wedge c > 0$.

Suppose $(a \square i_y) \wedge c = 0$. Then $c \leqq (a \square i_y)' = (a' \wedge i_x) \square i_y$. If now $a' \wedge i_x = 0$, $a = i_x$ and $(a \square i_y) \wedge c = i \wedge c = c = 0$. But then $c_x = 0$ and $a = 0$, which is a contradiction. If $a' \wedge i_x > 0$ then $a \leqq c_x \leqq a' \wedge i_x \leqq a'$ whence $a = 0$, also a contradiction.

$C30$. *If* p *is an atom in* $\mathfrak{B}$, *so are* p_x *and* p_y.

Suppose $0 < a < p_x$. By $C29$, $0 < (a \square i_y) \wedge p \leqq p$. Hence $(a \square i_y) \wedge p = p$ and $p \leqq a \square i_y$, $p_{\bar{x}} \leqq a$, a contradiction.

$C31$. If $0 < c$, and $0 < p \leqq c_x$, where p is a point, then $p = ((p \square i_y) \wedge c)_x$. Moreover, if q is a point in $(p \square i_y) \wedge c$, then $q_x = p$.

This follows from $C29$, $P3$, and $C4$, thus: $0 < ((p \square i_y) \wedge c)_x \leqq p \wedge c_x = p$.

3. Completion of projective algebras. We shall prove the following general theorem:

THEOREM 1. *If* $\mathfrak{B}$ *is a projective algebra, then* $\mathfrak{B}$ *is embeddable with preservation of (unrestricted) union, intersection, complement, x- and y-projections, and* $\square$*-products, in a projective algebra* $\mathfrak{C}$ *which is complete-ordered in the sense that every set of elements of* $\mathfrak{C}$ *has a l. u. b. and a g. l. b.*

Let $\mathfrak{B}$ be a projective algebra with elements $0 \leqq a \leqq i$. We recall that if $U(A)$, $L(A)$ are the sets of *all* upper and *all* lower bounds, respectively, of *all* elements of $A \subset \mathfrak{B}$, then $LU(A)$ is a closure operation.[1] Specifically, $A \subset LU(A)$, $LULU(A) = LU(A)$, (indeed one has $LUL(A) = L(A)$ and $ULU(A) = U(A)$), and $A \subset B$ implies $LU(A) \subset LU(B)$. The class $\mathfrak{C}$ of all $LU(A)$, $A \subset \mathfrak{B}$, is a complete ordered lattice which embeds $\mathfrak{B}$ under the correspondence $a \to LU(a) = L(a)$, $a \in \mathfrak{B}$, with preservation of unrestricted union and intersection. For any class of elements $LU(A_\alpha)$ of $\mathfrak{C}$, set intersection $\cap LU(A_\alpha)$ is effective as g. l. b. $\wedge LU(A_\alpha)$, and *closure* of set union $LU(\cup LU(A_\alpha))$ is effective as l. u. b. $\vee LU(A_\alpha)$. Note that $LU(i) = L(i) = \mathfrak{B}$ contains all $LU(A) \in \mathfrak{C}$ and is thus the identity of $\mathfrak{C}$. Also $LU(0) = L(0) = (0)$ is the zero of $\mathfrak{C}$. It is emphasized that while the elements of $\mathfrak{C}$ are sets, and the order is that of set inclusion, the boolean algebra $\mathfrak{C}$ is not the ordinary one of subsets of a set, in particular, union is *not* set union, and the zero of $\mathfrak{C}$ is *not* the null set.

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[1] G. Birkhoff, "Lattice theory," *American Mathematical Society Colloquium Publications*, vol. 25 (1940), p. 25; H. MacNeille, "Partially ordered sets," *Transactions of the American Mathematical Society*, vol. 42 (1937), pp. 416-460.

MacNeille [2] has shown that in $\mathfrak{C}$, $LU(A) \vee L(\text{all } a', a \in A) = LU(i)$, and $LU(A) \wedge L(\text{all } a') = LU(0)$, and that the correspondence $LU(A) \to L(\text{all } a')$ is a dual isomorphism of $\mathfrak{C}$ onto all of itself. From this it follows that $\mathfrak{C}$ is distributive and hence a boolean algebra, with $L(\text{all } a')$ effective as complement of $LU(A)$. The correspondence $a \to LU(a)$ obviously preserves complements, since $(LU(a))' = LU(a')$.

In this section we shall understand that if $A \subset \mathfrak{B}$, A_x is to mean the set of all a_x, $a \in A$. In $\mathfrak{C}$ define $(LU(A))_x = LU(A_x)$. We note that projection is well defined, i. e., $LU(A) = LU(B)$ implies $LU(A_x) = LU(B_x)$. It is sufficient to prove $A_x \subset LU(B_x)$. If $B = (0)$ we have $A \subset LU(A) = LU(0) = L(0) = (0)$; hence $A = (0)$, and $A_x = (0_x) = (0) \subset LU(B_x)$. Now let B contain at least one $b > 0$, and let $u \geqq B_x$. In particular, $u \geqq b_x > 0$, hence $u \wedge i_x \geqq b_x > 0$. Then $(u \wedge i_x) \square i_y) \geqq b_x \square b_y = b^* \geqq b$, all $b \in B$. But $U(A) = U(B)$, hence $(u \wedge i_x) \square i_y \geqq A$, and $u \geqq u \wedge i_x \geqq A_x$. Thus $A_x \subset LU(B_x)$. Similarly $B_x \subset LU(A_x)$. Moreover the correspondence $a \to LU(a)$ preserves projection, since $LU(a_x) = (LU(a))_x$.

Finally, in $\mathfrak{C}$ we define, for $(0) \neq LU(A) \subset LU(i_x)$, $(0) \neq LU(B) \subset LU(i_y)$, the direct product $LU(A) \square LU(B)$ as the (closed) set:

$L(\text{all } u \square v;\ u \leqq i_x,\ v \leqq i_y,\ u \square v \geqq \text{all } \dot{a} \square \dot{b},\ \dot{a} \in LU(A),\ \dot{b} \in LU(B))$.

The form of this definition is convenient in that it is clearly an element of $\mathfrak{C}$, since for any X, $L(X) = LUL(X)$; it is cumbersome, however, and we prove the following equivalences.

LEMMA 1. *For $LU(A) \subset LU(i_x)$, $LU(A) = L_x U_x(A)$ where the sub-x means that the operator is restricted to the elements $\leqq i_x$, that is to the elements of the sub-boolean algebra of all x-projections.*

Clearly $U(A) \supset U_x(A)$; hence $LU(A) \subset LU_x(A)$. But $LU_x(A) = L_x U_x(A)$. For $LU_x(A) \supset L_x U_x(A)$, and if $l \leqq U_x(A)$, then $l \leqq i_x$ since $i_x \geqq (A)$; (recall that $LU(i_x) = L(i_x) \supset LU(A) \supset A$). Hence $LU(A) \subset L_x U_x(A)$. Now let $l_x \leqq U_x(A)$, $u \geqq A$; then i_x, $u \geqq u \wedge i_x \geqq A$, hence $l_x \leqq u \wedge i_x \leqq u$.

LEMMA 2. $LU(A) \square LU(B) = L(\text{all } u \square v;\ u \in U_x(A),\ v \in (U_y(B))$.

For in the $\square$-definition u ranges over all $U_x L_x U_x(A) = U_x(A)$.

LEMMA 3. $LU(A) \square LU(B) = (\text{all } l;\ l \leqq \dot{a} \square \dot{b}$ *for some* $\dot{a} \in LU(A)$, $\dot{b} \in LU(B))$.

For let $l \in L(u \square v;\ u \in U_x(A),\ v \in U_y(B))$. Then $l \leqq u \square v$, $l_x \leqq \text{all } u$,

[2] H. MacNeille, *loc. cit.*

$l_y \leqq$ all v, $l_x \in L_xU_x(A)$, $l_y \in L_yU_y(B)$; hence $l_x = \dot{a}$, $l_y = \dot{b}$, and $l \leqq (l)^* = \dot{a} \square \dot{b}$. Conversely, let $l \leqq \dot{a} \square \dot{b}$, for $\dot{a} \in LU(A) = L_xU_x(A)$, $\dot{b} \in LU(B) = L_yU_y(B)$. Since $U_x(A) = U_xL_xU_x(A)$, $U_yL_yU_y(B)$, $l \leqq \dot{a} \square \dot{b} \leqq$ all $u \square v$. Thus $l \in L(\text{all } u \square v)$.

It is now clear that $a \to L(U(a))$ preserves $\square$-product, for $LU(a) \square LU(b) = LU(a \square b) = L(a \square b)$. For $LU(a) \square LU(b) = (l; l \leqq$ some $\dot{a} \square \dot{b}$, $\dot{a} \in LU(a)$, $\dot{b} \in LU(b)) = (l; l \leqq$ some $\dot{a} \square \dot{b}$, $\dot{a} \leqq a$, $\dot{b} \leqq b)$.

We proceed now to verify $P1$-7 in $\mathfrak{C}$.

$P1$. $(LU(A) \vee LU(B))_x = (LU(A))_x \vee (LU(B))_x$.

If $A = (0)$ or $B = (0)$, $P1$ is trivial. Using the definitions, we reduce $P1$ to $LU((LU(A) \cup LU(B))_x) = LU(LU(A_x) \cup LU(B))$. Operating on both sides with U and recalling that $ULU(A) = U(A)$, we have $U((LU(A) \cup LU(B))_x) = U(A_x) \cap U(B_x)$. Since $A \subset LU(A)$, $B \subset LU(B)$, A_x and B_x are in $(LU(A) \cup LU(B))_x$ and the left member is contained in the right. Now let $u \geqq A_x$, B_x, and $l \in LU(A)$. Then $u \triangle i_x \geqq A_x$, B_x, and $(u \wedge i_x) \square i_y \geqq a_x \square a_y \geqq a$ for all $a \in A$. Since at least one $a > 0$, $u \wedge i_x > 0$. Thus $l \leqq (u \wedge i_x) \square i_y$ and $l_x \leqq (u \wedge i_x) \leqq u$. Hence the left member is in the right.

$P2$. $(LU(i))_{xy} = (LU(i))_{yx}$ is a point in $\mathfrak{C}$; indeed both equal $LU(p_0) = (0, p_0)$.

For $(LU(i))_{xy} = LU(i_{xy}) = LU(p_0) = L(p_0) = (0, p_0)$, which is minimal over $LU(0)$.

$P3$. $(LU(A))_x = LU(0)$ if, and only if, $LU(A) = LU(0)$.

If $LU(A) = LU(0)$ clearly $(LU(A))_x = LU(0_x) = LU(0)$. Conversely if $LU(A_x) = (0) \supset A_x$, then $a_x = 0$ and $a = 0$ for all $a \in A$. Then $LU(A) = LU(0)$.

$P4$. $(LU(A))_{xx} = (LU(A))_x$.

The proof is trivial.

$P5$. For $(0) \neq LU(A) \subset LU(i_x)$ and $(0) \neq LU(B) \subset LU(i_y)$, the direct product has x-projection $LU(A)$, y-projection $LU(B)$, and is maximal in $\mathfrak{C}$ with these properties.

First, $LU((L(u \square v; u \square v \geqq$ all $\dot{a} \square \dot{b}$, $\dot{a} \in LU(A)$, $\dot{b} \in LU(B))_x) = LU(A)$. Let $l \leqq$ all $u \square v$, $u_0 \in U(A) = ULU(A)$, hence $u_0 \geqq$ all $\dot{a}$, and $u_0 \wedge i_x \geqq$ all $\dot{a}$, hence $u_0 \wedge i_x > 0$ since $LU(A) \neq (0)$. Then $(u_0 \wedge i_x) \square i_y \geqq$ all $\dot{a} \square \dot{b}$, and $l \leqq (u_0 \wedge i_x) \square i_y$. Thus $l_x \leqq u_0 \wedge i_x \leqq u_0$. Hence the left side is in the right. Now let $0 < a_0 \in A$, $0 < b_0 \in B$; then $a_0 \square b_0 \geqq$ all $u \square v$, $a_0 = (a_0 \square b_0)_x$, and $A \subset (L(u \square v))_x$.

Second, suppose $(LU(S))_x = LU(A)$, $(LU(S))_y = LU(B)$, and let $s \in S$. Then $s_x \in LU(A)$, $s_y \in LU(B)$, and $u \square v \geqq s_x \square s_y \geqq s$. Thus $S \subset L(u \square v) = LU(A) \square LU(B)$.

P6. $LU(i_x) \square LU(p_0) = LU(i_x)$.

This follows since $\square$-product is preserved under $a \to LU(a)$.

Before proving *P7*, we note the following:

LEMMA 4. *If* $S \subset LU(i_x)$, $u_1 \in LU(S)$, $u_0 \geqq S \square i_y$, *then* $u_1 \square i_y \leqq u_0$.

First, if $i_x \in S$, then $u_0 \geqq i_x \square i_y = i$, and the conclusion is trivial. Suppose $i_x \notin S$. We prove $u'_0 \leqq (u_1 \square i_y)' = (u'_1 \wedge i_x) \square i_y$. Since $u'_0 \leqq (u'_0)_x \square (u'_0)_y$, it is sufficient to prove $(u'_0)_x \leqq u'_1 \wedge i_x$, i. e., $(u'_0)_x \leqq u'_1$. But it is given that $u'_0 \leqq (S' \wedge i_x) \square i_y$; hence $(u'_0)_x \leqq S' \wedge i_x \leqq S'$. For if any $s' \wedge i_x = 0$, $s = i_x \in S$ contrary to our present hypothesis. But it is given that $u_1 \leqq U(S)$; hence $u'_1 \geqq (U(S))' = L(S')$. Thus $u'_1 \geqq (u'_0)_x$.

P7. $(LU(A) \vee LU(B)) \square LU(i_y)$
$= (LU(A) \square LU(i_y)) \vee (LU(B) \square LU(i_y))$.

If either $A = (0)$ or $B = (0)$, *P8* is trivially satisfied. Moreover, in any case, the right member is contained in the left by *C9* which follows from *P1-6*.

We remark that if X and Y are any two subsets of $\mathfrak{B}$, then $LU(X \cup Y) = LU(\text{all } x \vee y;\ x \in X, y \in Y)$.

Now note the equalities: $(LU(A) \vee LU(B)) \square LU(i_y) = LU(LU(A) \cup LU(B)) \square L(i_y) = LU(\text{all } \dot{a} \vee \dot{b};\ \dot{a} \in LU(A),\ \dot{b} \in LU(B)) \square L(i_y) = (\text{all } l \leqq \text{some } u \square v;\ u \in LU(\dot{a} \vee \dot{b}), v \in L(i_y)) = (\text{all } l \leqq \text{some } u \square i_y;\ u \in LU(\text{all } \dot{a} \vee \dot{b}))$.

We remark that $U(\text{all } l \leqq \text{some } \dot{a} \square i_y) = U(\text{all } \dot{a} \square i_y)$, and then note the equalities: $LU(A) \square LU(i_y) \vee (LU(B) \square LU(i_y)) = LU((\text{all } l \leqq \text{some } \dot{a} \square i_y) \cup (\text{all } l \leqq \text{some } \dot{b} \square i_y)) = L(U(l \leqq \text{some } \dot{a} \square i_y) \cap U(l \leqq \text{some } \dot{b} \square i_y)) = LU((\text{all } \dot{a} \square i_y) \cup (\text{all } \dot{b} \square i_y)) = LU(\text{all } a \square i_y \vee \dot{b} \square i_y) = LU(\text{all } (\dot{a} \vee \dot{b}) \square i_y)$.

We must therefore prove that $(\text{all } l \leqq \text{some } u \square i_y;\ u \in LU(\text{all } \dot{a} \vee \dot{b})) \subset LU(\text{all } (\dot{a} \vee \dot{b}) \square i_y)$. Let $l \leqq \text{some } u_1 \square i_y$, $u_1 \in LU(\text{all } \dot{a} \vee \dot{b})$, and $u_0 \geqq \text{all } (\dot{a} \vee \dot{b}) \square i_y$. Now use Lemma 4 with $S = (\text{all } \dot{a} \vee \dot{b})$. It follows that $u_1 \square i_y \leqq u_0$. Since $l \leqq u_1 \square i_y \leqq u_0$, the theorem is proved.

4. Representation theory. We shall prove a representation theorem for the case where $\mathfrak{B}$ is a projective algebra of *all* subsets of a set i of points p. We prove first two preliminary lemmas.

LEMMA 1. *If N is any cardinal number, there exists a (commutative) group containing exactly N elements.*

For a finite N we may use the cyclic group of order N. Now let N be infinite, and let V be a vector space of N basis elements over the galois field $GF(2)$. The elements of V may be considered as functions on a set of power N to the set $(0, 1)$ where addition is component-wise, mod 2. Denote by W the subgroup of all elements of V with only a finite number of 1-components, together with the zero element. W consists of the mutually exclusive subsets W_j of elements containing exactly j 1-components. Thus $W = \sum_0^\infty W_j$. The power of W is therefore $1 + \sum_1^\infty N^j = N$.

LEMMA 2. *If i is any (additive) group of elements p, the correspondence $p \to p + q$, q fixed, p arbitrary in i, defines a permutation π_q (one-one transformation) of i onto all of i, and the correspondence $q \to \pi_q$ is an isomorphism of i onto a group Π of permutations of the elements of i. In particular* (a) *the powers of Π and of i are equal.* (b) *for every p, $q \in i$ there exists a $\pi_s \in \Pi$ such that $\pi_s(p) = q$,* (c) *if $\pi_p(s_0) = \pi_q(s_0)$ for some $s_0 \in i$, then $\pi_p = \pi_q$.*

This is only a statement of the familiar Cayley representation of a group by permutations.

Now let i be a projective algebra of *all* subsets of a set i. We recall that $p_0 = i_x \wedge i_y$ so that all points of i are in one of the following disjoint classes:

(a) p_0, (b) $p \in i_x$, $p \neq p_0$, (c) $p \in i_y$, $p \neq p_0$, (d) $p \notin i_x \vee i_y$.

Moreover it is clear (C. 30) that the points in i_x are precisely all p_x where p ranges over i; similarly for i_y. Indeed, from C. 31, it follows that a_x is exactly the set of all p_x with p in a.

Let X be the class consisting of p_0 and of all pairs (p_x, q), p_x ranging over all points $\neq p_0$ in i_x, q over all points in i. Let Y be the class consisting of p_0 and all pairs (p_y, q), p_y ranging over all points $\neq p_0$ in i_y, q over all points in i.

Define the relation $p \sim q$ on points p, q of i to mean $p_x = q_x$, $p_y = q_y$. This is an equivalence relation splitting i into mutually exclusive sets (p) of points. In each class we distinguish a special representative $\bar{p}$ (Zermelo). The points in any class $(\bar{p})$ are thus all points of i of equal closure $\bar{p}_x \square \bar{p}_y$.

We define a group on the points of i, and set up the correspondence $q \to \pi_q$ of Lemma 2, and set up the following correspondence on points of i to subsets of $[X; Y]$.

(a) $p_0 \to [p_0; p_0]$.
(b) for $p \in i_x$, $p \neq p_0$, $p \to \{[(p_x, q); p_0], q \in i\}$.
(c) for $p \in i_y$, $p \neq p_0$, $p \to \{[p_0; (p_y, q)], q \in i\}$.
(d) for $p \notin i_x \vee i_y$, and $p \in (\bar{p})$, if
$p \neq \bar{p}$, $p \to \{[(\bar{p}_x, q); (\bar{p}_y, \pi_p(q))], q \in i\}$.

But

$$\bar{p} \to \{[(\bar{p}_x, q); (\bar{p}_y, \pi_{\bar{p}}(q))], q \in i\}$$

and
$\{[(\bar{p}_x, q); (\bar{p}_y, \pi_t(q))], q \in i, t$ an arbitrary point in the i-complement of $(\bar{p})\}$.

For any $a \leqq i$, we let $a \to A$, where A is the set of all images of all points of a under (a-d).

LEMMA 3. *If* $s \to [x; y]$ *and* $t \to [x; y]$ *under* (a-d) *then* $s = t$.

Different points of the type (b) map into different $[x; y]$, since the point itself appears as one of the coordinates, and similarly for type (c). If $[(\bar{p}_x, q); (\bar{p}_y, \pi_s(q))]$ is the map of two different points, they would be points with the same closure, and hence in the same $(\bar{p})$. But the pair of elements q, $\pi_s(q)$ can arise only from a unique s (Lemma 2c), and the two points would then both be p or both be $\bar{p}$, according to (d).

LEMMA 4. *Every point* $[x; y]$ *of* $[X; Y]$ *occurs as an image of some point* p *of* i.

For the points of $[X; Y]$ fall into the following classes:

(a) $[x; y] = [p_0; p_0]$ which is the map of p_0.
(b) $[x; y] = [(p_x, q); p_0]$ which is the map of $p_x \leqq i_x$, $p_x \neq p_0$.
(c) $[x; y] = [p_0; (p_y, q)]$ which is the map of $p_y \leqq i_y$, $p_y \neq p_0$.
(d) $[x; y] = [(p_x, q); (r_y, s)]$ $p_x \neq p_0$, $r_y \neq p_0$. In i, form the product $p_x \square r_y \neq 0$. There exists at least one point $t \leqq p_x \square r_y$ and $0 < t_x \leqq p_x$, $0 < t_y \leqq r_y$. Since $t > 0$, $t_x = p_x$, $t_y = r_y$. Hence $[x; y] = [(\bar{t}_x, q); (\bar{t}_y, s)]$, $t_x \neq p_0$, $t_y \neq p_0$. But there exists an element $n \in i$ (Lemma 2b) such that $\pi_n(q) = s$. Thus $[x; y] = [(\bar{t}_x, q); (\bar{t}_y, \pi_n(q)]$ is the map of some point in $(\bar{t})$.

THEOREM 2. *A projective algebra defined on all subsets of a set* i *of points* p *is isomorphic to a projective algebra of certain subsets of a direct product* $[X; Y]$, *with preservation of unit* $(i \to [X; Y])$, *union, intersection, complement, projection, and* $\square$*-product.*

The correspondence is that already defined. If now $a \to A$, $b \to B$,

$a \vee b \to S$, $a \wedge b \to P$, one readily verifies that $S = A \cup B$, $P = A \cap B$, since each member includes the other. For the latter equality one uses Lemma 3.

Now let $a \to A$, $a' \to B$. Then $B =$ complement of A in $[X; Y]$. Suppose $[x; y] \in B$, i. e., $s \to [x; y]$ where $s \in a'$. Then $[x; y] \in A'$, for if $[x; y] \in A$ one has $t \to [x; y]$, $t \in a$. By Lemma 3, $s = t$. Hence $B \subset A'$. Now let $[x; y] \in A'$. By Lemma 4, $s \to [x; y]$, for some $s \in i$. Hence $s \in a'$, and $[x; y] \in B$.

Next let $a \to A$, $a_x \to B$. Then $(A)_x = B$. Let $[x; p_0] \in A_x$. For some y, $[x; y] \in A$, hence $p \to [x; y]$ for $p \in a$. By examining the correspondences (a-d) one sees that $p \to [x; y]$ implies $p_x \to [x; p_0]$ in all cases. For example, if p is of the type (d), $p_x \in i_x$, and $p_x \neq p_0$ (for if so, $p \leqq i_y$, cf. (C. 18)). Thus $p_x \to [x; p_0]$ under (b). Hence in general $[x; p_0] \in B$. Similarly, if $a \to A$, $a_y \to C$, then $(A)_y = C$.

Finally, let $0 < a \leqq i_x$, $0 < b \leqq i_y$, and $a \to A$, $b \to B$, $a \square b \to C$. Then $A \square B = C$. For, let $[x; y] \in C$. Then $p \to [x; y]$, $p \in a \square b$, and $p_x \leqq a$, $p_y \leqq b$. But as pointed out above, $p_x \to [x; p_0]$ and $p_y \to [p_0; y]$, hence $[x; p_0] \in A$, $[p_0; y] \in B$. Thus $[x; y] \in A \square B$.

Conversely if $[x; y] \in A \square B$, $[x, p_0] \in A$, $[p_0; y] \in B$. Then for $p \leqq a$, $q \leqq b$, $p \to [x; p_0]$, $q \to [p_0; y]$. But then $0 < p \square q \leqq a \square b$, and there is at least one point $t \leqq p \square q \leqq a \square b$. Thus $t_x = p$, $t_y = q$, and $t \to [x; y]$. Hence $[x; y] \in C$.

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CORRESPONDING TYPE CONTINUED FRACTIONS.*

By EVELYN FRANK.[1]

1. Introduction. This paper is concerned with a development of properties of *corresponding type continued fractions*

$$1+\frac{a_1z^{\alpha_1}}{1}+\frac{a_2z^{\alpha_2}}{1}+\frac{a_3z^{\alpha_3}}{1}+\cdots, \tag{1.1}$$

in which the a_p are complex numbers and the α_p are positive integers. For the sake of brevity, we shall call (1.1) a *C-fraction*.

It has been shown by Leighton and Scott [1][2] that an arbitrary power series $P(z)=1+c_1z+c_2z^2+\cdots$ can be expanded into a uniquely determined C-fraction, which terminates if, and only if, the power series represents a rational function of z. Scott and Wall [4] investigated the relationship of the C-fraction to the table of Padé approximants for $P(z)$. They called the C-fraction and its power series *regular* if all its approximants are Padé approximants, and determined a certain class of regular C-fractions.

In the present paper, we continue this investigation. The principal results may be summarized as follows:

A. Algorithm for expanding an arbitrary power series into a C-fraction. We find that there is an algorithm for expanding a power series into a C-fraction which is analogous to that given by Wall [6] for J-fractions. This gives an essential simplification of the expansion problem (cf. [1], p. 598; [4], p. 329), inasmuch as *all long division is eliminated* (**2**).

B. The Padé table. If in the Padé table for a power series $P(z)$ there are two equal approximants, then there must be a square *block* of $(r+1)^2$ equal approximants. Padé ([2], p. 429) determined necessary and sufficient conditions for the *order* r of a block to be zero. We have extended this theorem to cover *arbitrary* orders (Theorem 3.1).

C. Regular C-fractions. From a consideration of the geometrical arrangement of the blocks in the Padé table and their relationship to the exponents

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[2] Numbers in square brackets refer to the bibliography.

α_p in the C-fraction, we obtain a new formulation of the condition for the C-fraction to be regular (Theorem 4.1). In particular, we obtain the class of *α-regular* C-fractions in which the condition for regularity depends only upon the exponents α_p (Theorem 4.2).

D. Characterization of regular power series in terms of their coefficients. It is well known that a power series is semi-normal, i.e., $\alpha_p = 1$, if, and only if, certain determinants formed from the coefficients of the power series are different from zero. We extend this theorem to cover the class of α-regular power series, and indicate how an analogous theorem may be obtained for regular power series (**5**).

E. Some transformations of C-fractions. A remarkable example of a C-fraction was given by Ramanujan [3], namely,

$$1+\frac{z}{1}+\frac{z^2}{1}+\frac{z^3}{1}+\frac{z^4}{1}+\cdots.$$

This is identically equal to

$$1+z-\frac{z^3}{1}+\frac{z^2}{1}-\frac{z}{1}+\frac{z}{1}-\frac{z^3}{1}+\frac{z^3}{1}-\frac{z^2}{1}+\frac{z^2}{1}-\frac{z^4}{1}+\frac{z^4}{1}-\cdots.$$

We find that this transformation of the Ramanujan C-fraction can be carried out for all C-fractions for which $\alpha_2+\alpha_4+\cdots+\alpha_{2p+2} > \alpha_3+\alpha_5+\cdots+\alpha_{2p+1} > \alpha_2+\alpha_4+\cdots+\alpha_{2p}$ (Theorem 6.1). We also obtain a number of other transformations (Theorems 6.2, 6.4, 6.5, and 6.6).

2. Algorithm for expanding a power series into a C-fraction. The following theorem can be used to obtain the C-fraction for an arbitrary power series, and can also be used to expand a C-fraction into a power series.

THEOREM 2.1. *Let $P(z) = 1 + c_1 z + c_2 z^2 + \cdots$ be an arbitrary formal power series, and determine polynomials $B_p(z) = \beta_0^{(p)} + \beta_1^{(p)} z + \beta_2^{(p)} z^2 + \cdots$, numbers $a_p \neq 0$, and positive integers α_p, $p = 1, 2, \cdots$, by means of the recurrence formulas*

$$(2.1)\quad \beta_0^{(0)} = \beta_0^{(1)} = 1, \quad \beta_p^{(0)} = \beta_p^{(1)} = 0 \text{ for } p > 0; \quad \alpha_0 = 0;$$

$$(2.2)\quad (c_n, c_{n-1}, c_{n-2}, \cdots)\begin{pmatrix}\beta_0^{(p)}\\ \beta_1^{(p)}\\ \beta_2^{(p)}\\ \cdot\\ \cdot\\ \cdot\end{pmatrix} = \begin{cases} 0 \text{ if } \alpha_0+\alpha_1+\cdots+\alpha_p < n < \alpha_0+\alpha_1+\cdots+\alpha_{p+1}, \\ (-1)^p a_1 a_2 \cdots a_{p+1} \text{ if } n = \alpha_0+\alpha_1+\cdots+\alpha_{p+1}; \\ (p = 0, 1, 2, \cdots); \end{cases}$$

$$(2.3)\quad B_{p+1}(z) = B_p(z) + a_{p+1} z^{\alpha_{p+1}} B_{p-1}(z), \qquad (p = 1, 2, 3, \cdots).$$

Then

$$1+\frac{a_1z^{\alpha_1}}{1}+\frac{a_2z^{\alpha_2}}{1}+\frac{a_3z^{\alpha_3}}{1}+\cdots \tag{2.4}$$

is the C-fraction for $P(z)$. *It terminates if, and only if,* $P(z)$ *is a rational function of* z, *in which case there is a value* p' *of* p *such that the left-hand member of* (2.2) *is* 0 *for all* $n > \alpha_0+\alpha_1+\cdots+\alpha_{p'}$. *The polynomials* $B_p(z)$ *are the denominators of* (2.4). *The numerators are* $A_p(z)=\gamma_0^{(p)}+\gamma_1^{(p)}z+\cdots+\gamma_s^{(p)}z^s$ *where* $s=s_p$ *is at most the sum of the* $[(p+1)/2]$ *largest integers in the set* $\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_p$, *and, if* $c_0=1$,

$$(\gamma_0^{(p)}, \gamma_1^{(p)}, \gamma_2^{(p)}, \cdots) = (\beta_0^{(p)}, \beta_1^{(p)}, \beta_2^{(p)}, \cdots)\begin{pmatrix} c_0, c_1, c_2, \cdots, c_s, 0, \cdots \\ 0, c_0, c_1, \cdots, c_{s-1}, 0, \cdots \\ 0, 0, c_0, \cdots, c_{s-2}, 0, \cdots \\ \cdots\cdots\cdots\cdots \end{pmatrix}. \tag{2.5}$$

Proof. Suppose first that $P(z)$ is not a rational function of z, so that the recurrent process obviously cannot terminate, and the infinite C-fraction (2.4) can be constructed. From (2.1), (2.3) it follows that $B_p(z)$ is its p-th denominator. Its numerators are given by the recurrence formulas

$$A_0(z)=1,\ A_1(z)=1+a_1z^{\alpha_1},\ A_{p+1}(z)=A_p(z)+a_{p+1}z^{\alpha_{p+1}}A_{p-1}(z), \quad (p=1,2,\cdots). \tag{2.6}$$

One may verify at once that $P(z)B_0(z)-A_0(z)=a_1z^{\alpha_1}+\cdots$. Using induction, we suppose that

$$P(z)B_p(z)-A_p(z)=(-1)^p a_1a_2\cdots a_{p+1}z^{\alpha_1+\alpha_2+\cdots+\alpha_{p+1}}+\cdots \tag{2.7}$$

for $p=0,1,\cdots,m$, and shall prove it for $p=m+1$. We have

$$\begin{aligned} P(z)B_{m+1}(z)-A_{m+1}(z) &= P(z)[B_m(z)+a_{m+1}z^{\alpha_{m+1}}B_{m-1}(z)] \\ &\quad -[A_m(z)+a_{m+1}z^{\alpha_{m+1}}A_{m-1}(z)] = [P(z)B_m(z)-A_m(z)] \\ &\quad +a_{m+1}z^{\alpha_{m+1}}[P(z)B_{m-1}(z)-A_{m-1}(z)] = (z^{1+\alpha_1+\alpha_2+\cdots+\alpha_{m+1}}), \end{aligned}$$

where (z^k) denotes a power series containing no lower power of z than the k-th power. Since, as may be easily seen, the degree of $A_{m+1}(z)$ is at most the sum of the $[(m+2)/2]$ largest integers in $\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_{m+1}$, and is therefore less than $\alpha_0+\alpha_1+\alpha_2+\cdots+\alpha_{m+1}$, it follows that the terms of degree greater than this value in $P(z)B_{m+1}(z)-A_{m+1}(z)$ are the same as those in $P(z)B_{m+1}(z)$. By (2.2) the first such term which occurs is

$$(-1)^{m+1}a_1a_2\cdots a_{m+2}z^{\alpha_1+\alpha_2+\cdots+\alpha_{m+2}},$$

and consequently (2.7) holds for $p = m + 1$, as was to be proved. The formula (2.5) may be obtained by equating coefficients of $z^0, z^1, \cdots, z^s$ on either side of (2.7).

If $P(z)$ is a rational function of z, the recurrent process must terminate with some $p = p'$, for such a function cannot have an infinite C-fraction. If, in the preceding induction, we assume that $m < p'$, it follows that (2.7) holds for $p = 0, 1, \cdots, p' - 1$, while $P(z)B_{p'}(z) - A_{p'}(z) \equiv 0$.

To illustrate the theorem, let $P(z) = 1 - z^2 + z^4$, a polynomial. Then we have $a_1 = -1$, $\alpha_1 = 2$, $a_2 = 1$, $\alpha_2 = 2$, so that $B_2(z) = 1 + z^2$. From (2.2) with $p = 2$, $n > \alpha_1 + \alpha_2 = 4$, we get

$$(0, 1, 0)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0, \qquad (0, 0, 1)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 1 = a_1 a_2 a_3 = -a_3,$$

so that $a_3 = -1$, $\alpha_3 = 2$, $B_3 = 1$, and the process terminates. Hence

$$1 - z^2 + z^4 = 1 - \frac{z^2}{1} + \frac{z^2}{1} - \frac{z^2}{1}.$$

The formulas of Theorem 2.1 may also be used to obtain the power series expansion of a given C-fraction.

3. A theorem on the Padé table. To the point (m, n) with non-negative integral coordinates m and n, in the cartesian plane, we let correspond the *Padé approximant* $f_{m,n}(z) = N_{m,n}(z)/D_{m,n}(z)$ of the power series $P(z) = c_0 + c_1 z + c_2 z^2 + \cdots$ $(c_0 \neq 0)$, determined uniquely by the following conditions: $D_{m,n} \not\equiv 0$ is of degree $\leq m$, $N_{m,n}(z)$ is of degree $\leq n$, $PD_{m,n} - N_{m,n} = (z^{m+n+1})$. We shall suppose that the positive x-axis extends downward and the positive y-axis extends to the right. We may also regard $f_{m,n}(z)$ as occupying the *square* $[m, n]$ with vertices (m, n), $(m, n + 1)$, $(m + 1, n)$, $(m + 1, n + 1)$. This geometrical configuration is called the *Padé table* for $P(z)$ (cf., for instance, [2], chapter 10). A particular rational fraction may occur more than once as a Padé approximant in the table, but in any case it fills a *block* of squares

$$[m + p, n + q], \qquad (p, q = 0, 1, \cdots, r), \tag{3.1}$$

and occurs nowhere else in the table ([2], p. 427). We call r the *order* of the approximant and of the block of squares which it occupies. The theorem which we shall now prove gives necessary and sufficient conditions upon the coefficients of $P(z)$ for the Padé table to contain this block of order r.

THEOREM 3.1. *Let* $P(z) = c_0 + c_1 z + c_2 z^2 + \cdots$, $c_0 \neq 0$, *be an arbitrary power series, and put*

$$\Delta_{m,n} = \begin{vmatrix} c_{n-m}, & c_{n-m+1}, & \cdots, & c_n \\ c_{n-m+1}, & c_{n-m+2}, & \cdots, & c_{n+1} \\ \cdot & \cdot & \cdot & \cdot \\ c_n, & c_{n+1}, & \cdots, & c_{n+m} \end{vmatrix}, \qquad (m, n = 0, 1, 2, \cdots), \tag{3.2}$$

where we agree to set $c_p = 0$ *if* $p < 0$. *It will be convenient to define* $\Delta_{-1,n} = 1$, $(n = 0, 1, 2, \cdots)$. *In order for the Padé table for* $P(z)$ *to contain the block* (3.1) *of order* r, *it is necessary and sufficient that all of the following conditions hold:*

$$\begin{array}{lll} \text{(i)}\ \Delta_{t-1,s} \neq 0, & \text{(ii)}\ \Delta_{t-1,s+1} \neq 0, & \text{(iii)}\ \Delta_{t,s} \neq 0, \\ \text{(iv)}\ \Delta_{t+k,s+k+1} = 0,\ (k = 0, 1, \cdots, r-1), & & \text{(v)}\ \Delta_{t+r,s+r+1} \neq 0. \end{array} \tag{3.3}$$

Proof. It is clearly necessary and sufficient for the Padé table to contain the given block that

$$\begin{array}{lll} \text{(i')}\ f_{t-1,s-1} \neq f_{t,s}, & \text{(ii')}\ f_{t-1,s} \neq f_{t,s}, & \text{(iii')}\ f_{t,s-1} \neq f_{t,s}, \\ \text{(iv')}\ f_{t+p,s+p} = f_{t,s},\ (p = 1, 2, \cdots, r), & & \text{(v')}\ f_{t+r+1,s+r+1} \neq f_{t,s}. \end{array} \tag{3.4}$$

We shall show first that (3.3) is *necessary* for (3.4). Let $N = \alpha_0 + a_1 z + \cdots + a_s z^s$, $D = \beta_0 + \beta_1 z + \cdots + \beta_t z^t$,

$$\Gamma_{m,n} = \begin{pmatrix} c_n, & c_{n-1}, & \cdots, & c_{n-m} \\ c_{n+1}, & c_n, & \cdots, & c_{n-m+1} \\ \cdot & \cdot & \cdot & \cdot \\ c_{n+m-1}, & c_{n+m-2}, & \cdots, & c_{n-1} \end{pmatrix}, \qquad \alpha = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}, \qquad \beta = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}.$$

Then the condition for N/D to be the Padé approximant $f_{t,s}(z)$ of $P(z)$ is:

$$\Gamma_{s+1,0}\beta = \alpha, \qquad \Gamma_{t,s+1}\beta = 0, \quad \beta \neq 0. \tag{3.5}$$

If $\Delta_{t-1,s} = 0$, then (3.5) has a solution with $\beta_0 = 0$, $\alpha_0 = 0$, so that $f_{t-1,s-1} = f_{t,s}$; if $\Delta_{t-1,s+1} = 0$, there is a solution with $\beta_t = 0$, so that $f_{t-1,s} = f_{t,s}$; if $\Delta_{t,s} = 0$, there is a solution with $\alpha_s = 0$, so that $f_{t,s-1} = f_{t,s}$. Thus (i), (ii), (iii) are necessary for (i'), (ii'), (iii'), respectively. In order for (iv') to hold, it is necessary that $z^p N / z^p D$ be a Padé approximant for $p = 1, 2, \cdots, r$, i.e., that $Pz^p D - z^p N$ contain no lower power of z than the $(t + s + 2p + 1)$-th, for $p = 1, 2, \cdots, r$. If $\beta^{(p)}$ is the matrix obtained

from β by inserting p zeros above β_0, this condition can be written $\Gamma_{t+p,s+p+1}\beta^{(p)} = 0$, $(p = 1, 2, \cdots, r)$. Inasmuch as $\beta^{(p)} \neq 0$, this implies (iv). Finally, (v) is necessary for (v'), for the same reason that (i) is necessary for (i').

We shall now prove that (3.3) is *sufficient* for (3.4). By (i) the system (3.5) has a solution α, β, and every other solution has the form $c\alpha$, $c\beta$, where c is a constant. Moreover, $\beta_0 \neq 0$. Therefore, the fraction N/D is *irreducible*. For, if $N/D \equiv N^*/D^*$ where N^* and D^* are of lower degree by $h > 0$ than N and D, then $Pz^hD^* - z^hN^*$ would contain no lower power of z than the $(s + t + 1)$-th ([2], p. 423). Hence, the polynomials z^hD^* and z^hN^* could be determined by solving (3.5). But in this solution we would have $\beta_0 = 0$, which is impossible. Condition (ii) implies that $\beta_t \neq 0$. This together with the fact that N/D is irreducible shows that (ii') holds. Condition (iii) implies that $\alpha_s \neq 0$, so that (iii') holds. To prove (iv'), let $f_{t+p,s+p} = N^{(p)}/D^{(p)}$, and let $\alpha^{(p)}$, $\beta^{(p)}$ be the one-column matrices of the coefficients of $N^{(p)}$ and $D^{(p)}$. Since $PD^{(p)} - N^{(p)} = (z^{s+t+2p+1})$, we have

$$(3.6) \qquad \Gamma_{s+p+1,0}\beta^{(p)} = \alpha^{(p)}, \quad \Gamma_{t+p,s+p+1}\beta^{(p)} = 0, \qquad (p = 1, 2, \cdots, r).$$

Since $\Delta_{t,s+1} = 0$, we see that for $p = 1$ (3.6) has the solution $\beta_0^{(1)} = 0$, $\beta_k^{(1)} = \beta_{k-1}$, $(k = 1, 2, \cdots, t + 1)$, $\alpha_0^{(1)} = 0$, $\alpha_k^{(1)} = \alpha_{k-1}$, $(k = 1, 2, \cdots, s + 1)$. Hence it follows that $N^{(1)}/D^{(1)} \equiv N/D$, and (iv') holds for $p = 1$. Then, since $\Delta_{t+1,s+2} = 0$, it follows that, for $p = 2$, (3.6) has the solution $\beta_0^{(2)} = 0$, $\beta_k^{(2)} = \beta_{k-1}^{(1)}$, $(k = 1, 2, \cdots, t + 2)$, $\alpha_0^{(2)} = 0$, $\alpha_k^{(2)} = \alpha_{k-1}^{(1)}$, $(k = 1, 2, \cdots, s + 2)$, where $\alpha^{(1)}$, $\beta^{(1)} \neq 0$ is a solution of (3.6) for $p = 1$. Hence $N^{(2)}/D^{(2)} \equiv N^{(1)}/D^{(1)} \equiv N/D$, so that (iv') holds for $p = 2$. Continuing this process we conclude that (iv') holds for $p = 1, 2, \cdots, r$. Finally, since $\Delta_{t+r,s+r+1} \neq 0$, $PD^{(r)} - N^{(r)}$ actually contains the $(t + s + 2r + 1)$-th power of z, so that (v') holds. For a like reason, (i) implies (i').

This completes the proof of Theorem 3.1.

The preceding proof contains the proof of the following theorem.

THEOREM 3.2. *In the Padé table for $P(z)$, we have for some integers p, q, r:*

$$(3.7) \quad \text{(i) } f_{q-1,p-1} \neq f_{q,p}, \quad \text{(ii) } f_{q+k,p+k} = f_{q,p}, \ (k = 1, 2, \cdots, r), \quad \text{(iii) } f_{q+r+1,p+r+1} \neq f_{q,p}$$

if, and only if,

$$(3.8) \quad \text{(i')} \ \Delta_{q-1,p} \neq 0, \quad \text{(ii')} \ \Delta_{q+k,p+k+1} = 0, \ (k = 0, 1, \cdots, r - 1), \quad \text{(iii')} \ \Delta_{q+r,p+r+1} \neq 0.$$

4. Regular C-fractions. The C-fraction (1.1) and its power series $P(z) = 1 + c_1z + c_2z^2 + \cdots$ are called *regular* if every approximant

$A_p(z)/B_p(z)$ is a Padé approximant for $P(z)$. We shall first reformulate some results of Scott and Wall [4] on the Padé table for a regular power series. The novel aspect of this formulation consists in the introduction of the *α-polygon*. We shall then characterize regular C-fractions in terms of geometrical properties of the Padé table, and then (**5**) in terms of the coefficients of the power series.

If s_p and t_p are the degrees of $A_p(z)$ and $B_p(z)$, respectively, then $P(z)$ is regular if, and only if, for every p, $P(z)B_p(z) - A_p(z) = (z^{s_p+t_p+1})$ or if, and only if, there is an integer r_p such that

$$(4.1)\quad s_p + t_p + r_p + 1 = \alpha_1 + \alpha_2 + \cdots + \alpha_{p+1},\ r_p \geq 0,\ (p = 0, 1, 2, \cdots).$$

Since the approximant is irreducible, r_p is its order [2]. Let Q_p denote the block of order r_p whose squares are occupied by $A_p(z)/B_p(z)$. The coordinates of its vertices are (t_p, s_p), $(t_p, s_p + r_p + 1)$, $(t_p + r_p + 1, s_p)$, $(t_p + r_p + 1, s_p + r_p + 1)$. For all points on the diagonal connecting the second and third of these vertices, the sum of the coordinates is $s_p + t_p + r_p + 1$. By (4.1), this sum increases with increasing p. Hence, as Q_p and Q_{p+1} do not overlap, it follows that for any given p either

$$(4.2)\quad s_{p+1} > s_p + r_p,$$

or

$$(4.3)\quad t_{p+1} > t_p + r_p.$$

By (2.3), (2.6), it follows that

$$(4.4)\quad s_{p+1} = s_{p-1} + \alpha_{p+1},$$

or

$$(4.5)\quad t_{p+1} = t_{p-1} + \alpha_{p+1},$$

according as (4.2) or (4.3) holds, respectively. Hence,

$$(4.6)\quad s_{p+1} - (s_p + r_p) = t_p - (t_{p-1} + r_{p-1}) > 0,$$

or

$$(4.7)\quad t_{p+1} - (t_p + r_p) = s_p - (s_{p-1} + r_{p-1}) > 0,$$

respectively. One may then readily show by mathematical induction that

$$(4.8)\quad \begin{aligned} &s_{2p+1} > s_{2p} + r_{2p}, \quad && s_{2p+2} \leq s_{2p+1} + r_{2p+1}, \\ &t_{2p+1} \leq t_{2p} + r_{2p}, && t_{2p+2} > t_{2p+1} + r_{2p+1}, \qquad (p = 0, 1, 2, \cdots). \end{aligned}$$

Hence we find, by (4.4), (4.5) that

$$(4.9)\quad \begin{aligned} s_{2p-1} &= \alpha_1 + \alpha_3 + \cdots + \alpha_{2p-1}, \\ t_{2p} &= \alpha_2 + \alpha_4 + \cdots + \alpha_{2p}, \qquad (p = 1, 2, 3, \cdots). \end{aligned}$$

By (4. 1) it follows that

$$(4.10)\qquad \begin{aligned} 1 + s_{2p} \quad + r_{2p} \quad &= \alpha_1 + \alpha_3 + \cdots + \alpha_{2p+1}, \\ 1 + t_{2p-1} + r_{2p-1} &= \alpha_2 + \alpha_4 + \cdots + \alpha_{2p}, \end{aligned}$$

or

$$(4.11)\qquad \begin{aligned} s_{2p+1} &= s_{2p} \quad + r_{2p} \quad + 1, \\ t_{2p} \quad &= t_{2p-1} + r_{2p-1} + 1. \end{aligned}$$

These results have a simple geometrical interpretation in the Padé table. We first draw the α-polygon $OV_0V_1V_2\cdots$ with vertices $O = (0, 0)$, $V_0 = (0, \alpha_1)$, $V_{2p-1} = (\alpha_2 + \alpha_4 + \cdots + \alpha_{2p}, \alpha_1 + \alpha_3 + \cdots + \alpha_{2p-1})$, $V_{2p} = (\alpha_2 + \alpha_4 + \cdots + \alpha_{2p}, \alpha_1 + \alpha_3 + \cdots + \alpha_{2p+1})$, $(p = 1, 2, 3, \cdots)$. By (4. 9), (4. 10), the vertices of Q_{2p} are

$$W_{2p} = (\alpha_2 + \alpha_4 + \cdots + \alpha_{2p}, s_{2p}), \quad V_{2p},$$
$$T_{2p} = (t_{2p} + r_{2p} + 1, s_{2p}), \qquad U_{2p} = (t_{2p} + r_{2p} + 1, \alpha_1 + \alpha_3 + \cdots + \alpha_{2p+1}),$$

and the vertices of Q_{2p+1} are

$$U_{2p+1} = (t_{2p+1}, \alpha_1 + \alpha_3 + \cdots + \alpha_{2p+1}), \quad T_{2p+1} = (t_{2p+1}, s_{2p+1} + r_{2p+1} + 1),$$
$$V_{2p+1}, \qquad W_{2p+1} = (\alpha_2 + \alpha_4 + \cdots + \alpha_{2p}, s_{2p+1} + r_{2p+1} + 1).$$

Thus, the α-polygon forms part of the boundary of every square Q_p.

By (4. 8) we see that Q_p and Q_{p+1} have a line segment D_p of the α-polygon as common boundary (Fig. 1).

Let $L(\omega)$ denote the straight line whose equation is $y = x + \omega$. Then it is clear that there exist one or more integers ω_p such that the lines $L(\omega_p)$ and $L(1 + \omega_p)$ cut the line segment D_{2p}. The principal diagonal of Q_{2p} lies on or below $L(\omega_p)$, and the principal diagonal of Q_{2p+1} lies on or above $L(1 + \omega_p)$. Hence we see that W_{2p} must lie on or below $L(\omega_p)$, U_{2p+1} must lie on or above $L(1 + \omega_p)$, V_{2p} must lie on or above $L(1 + \omega_p)$, and V_{2p+1} must lie on or below $L(\omega_p)$. These statements are equivalent to the following inequalities:

$$(4.12)\qquad \begin{aligned} &\text{(i)} \quad s_{2p} \leq t_{2p} + \omega_p, \\ &\text{(ii)} \quad s_{2p+1} \geq t_{2p+1} + \omega_p + 1, \\ &\text{(iii)} \quad s_{2p+1} \geq t_{2p} + \omega_p + 1, \\ &\text{(iv)} \quad s_{2p+1} \leq t_{2p+2} + \omega_p, \qquad (p = 0, 1, 2, \cdots). \end{aligned}$$

We shall now prove that the *necessary* conditions (4. 12) and (4. 9) are *sufficient* for regularity. We suppose that (4. 9) holds, and that integers $\omega_0, \omega_1, \cdots$ exist such that (4. 12) holds. Then $s_{2p} + t_{2p} = s_{2p} + \alpha_2 + \alpha_4$

$+\cdots+\alpha_{2p} \leq t_{2p}+\omega_p+\alpha_2+\alpha_4+\cdots+\alpha_{2p} \leq s_{2p+1}-1+\alpha_2+\alpha_4$ $+\cdots+\alpha_{2p}$, or $s_{2p}+t_{2p}+1 \leq \alpha_1+\alpha_2+\cdots+\cdots+\alpha_{2p+1}$. Similarly, $s_{2p+1}+t_{2p+1}=\alpha_1+\alpha_3+\cdots+\alpha_{2p+1}+t_{2p+1} \leq \alpha_1+\alpha_3+\cdots+\alpha_{2p+1}+s_{2p+1}$

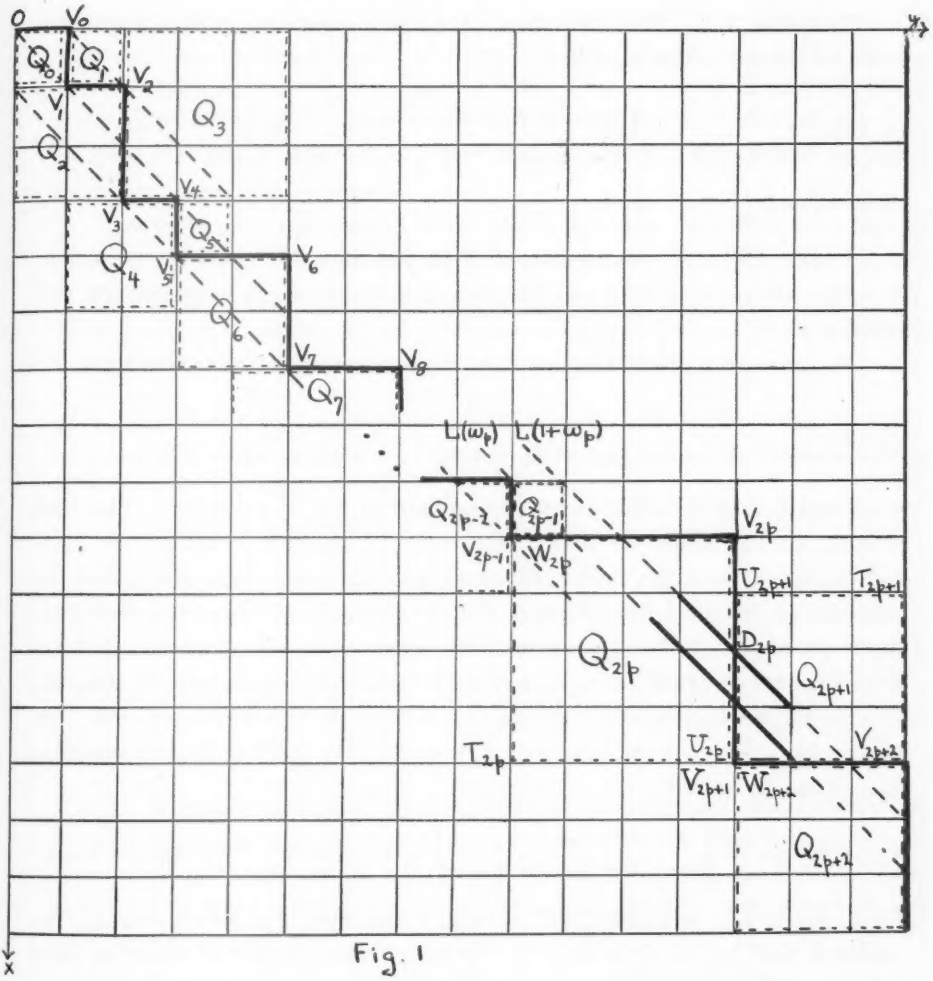


Fig. 1

$-\omega_p-1 \leq \alpha_1+\alpha_3+\cdots+\alpha_{2p+1}+t_{2p+2}-1$, or $s_{2p+1}+t_{2p+1}+1 \leq \alpha_1+\alpha_2$ $+\cdots+\alpha_{2p+2}$. Consequently, (4. 1) holds, and hence the C-fraction is regular.

We have proved the following theorem:

THEOREM 4. 1. *The C-fraction* (1. 1) *is regular if, and only if,* (4. 9) *and* (4. 12) *hold, where* $\omega_0, \omega_1, \cdots$ *are integers.*

If (4. 9) holds, and there is an integer $\omega \geq 0$ such that the straight lines

$L(\omega)$ and $L(1+\omega)$ intersect all the line segments D_p, $(p=0,1,2,\cdots)$, then the C-fraction and its power series are called *α-regular*. We shall prove the following theorem:

THEOREM 4.2. *The C-fraction* (1.1) *is α-regular if and only if there exists an integer $\omega \geq 0$ such that*

$$(4.13)\quad \begin{aligned} &\alpha_1+\alpha_3+\cdots+\alpha_{2p+1} \geq \alpha_0+\alpha_2+\alpha_4+\cdots+\alpha_{2p}+\omega+1,\\ &\alpha_2+\alpha_4+\cdots+\alpha_{2p+2} \geq \alpha_1+\alpha_3+\cdots+\alpha_{2p+1}-\omega, \quad (p=0,1,2,\cdots), \end{aligned}$$

where $\alpha_0=0$.

Proof. If (1.1) is α-regular, then (4.12) holds with $\omega=\omega_p$, $(p=0,1,2,\cdots)$. Hence, by (4.9) and (4.12) (iii), (iv), we conclude that (4.13) holds.

If, conversely, (4.13) holds, then we shall prove by induction that

$$(4.14)\quad \begin{aligned} &s_{2p}-\omega \leq t_{2p}=\alpha_0+\alpha_2+\alpha_4+\cdots+\alpha_{2p}, \qquad (p=0,1,2,\cdots),\\ &\omega+1+t_{2p-1} \leq s_{2p-1}=\alpha_1+\alpha_3+\cdots+\alpha_{2p-1}, \quad (p=1,2,3,\cdots), \end{aligned}$$

from which it will follow immediately that (1.1) is α-regular. The first relation (4.14) holds for $p=0$ since $\omega \geq 0$. Since $s_1=\alpha_1$, $t_1=0$, $s_2 \leq \max(\alpha_1,\alpha_2)$, $t_2=\alpha_2$, we readily verify (4.14) for $p=1$. Assuming that the relations (4.14) hold for $p \leq n$, $n \geq 1$, we shall prove them for $p=n+1$. From (2.6) we see that $s_{2n+1}=\max(s_{2n}, s_{2n-1}+\alpha_{2n+1})$ if $s_{2n} \neq s_{2n-1}+\alpha_{2n+1}$. But, by our assumption, $s_{2n} \leq \omega+\alpha_2+\alpha_4+\cdots+\alpha_{2n}$ and, by (4.13), $s_{2n-1}+\alpha_{2n+1}=\alpha_1+\alpha_3+\cdots+\alpha_{2n+1} > \alpha_2+\alpha_4+\cdots+\alpha_{2n}+\omega$, so that $s_{2n+1}=\alpha_{2n+1}+s_{2n-1}=\alpha_1+\alpha_3+\cdots+\alpha_{2n+1}$. By (2.3), $t_{2n+1} \leq \max(t_{2n}, t_{2n-1}+\alpha_{2n+1})$, so that

$$\omega+1+t_{2n+1} \leq \left\{\begin{matrix} \omega+1+t_{2n} \\ \omega+1+t_{2n-1}+\alpha_{2n+1} \end{matrix}\right\} \leq \alpha_1+\alpha_3+\cdots+\alpha_{2n+1}=s_{2n+1}.$$

Next, since $t_{2n+2}=\max(t_{2n+1}, t_{2n}+\alpha_{2n+2})$ provided $t_{2n+1} \neq t_{2n}+\alpha_{2n+2}$, we conclude that $t_{2n+2}=\alpha_2+\alpha_4+\cdots+\alpha_{2n+2}$, inasmuch as, by what we have just proved and (4.13), $t_{2n+1} \leq \alpha_1+\alpha_3+\cdots+\alpha_{2n+1}-\omega-1 < \alpha_2+\alpha_4+\cdots+\alpha_{2n+2}=t_{2n}+\alpha_{2n+2}$. Finally, we have

$$s_{2n+2} \leq \left\{\begin{matrix} s_{2n+1} \\ s_{2n}+\alpha_{2n+2} \end{matrix}\right\} \leq \alpha_2+\alpha_4+\cdots+\alpha_{2n+2}+\omega=t_{2n+2}+\omega.$$

The proof of Theorem 4.2 is now complete.

In general, the regularity of (1.1) depends upon the coefficients a_p as well as upon the exponents α_p. For example, the C-fraction

$$1+\frac{z}{1}-\frac{z}{1}+\frac{z}{1}-\frac{z^2}{1}-\frac{z}{1}+\frac{z}{1}+\frac{z^2}{1}-\frac{z^2}{1}+\frac{z^2}{1}+\frac{z^2}{1}+\frac{z^2}{1}+\cdots$$

is regular (Fig. 1), but if the minus signs are replaced by plus signs, the resulting C-fraction is not regular. The conditions (4.13) involve *only* the exponents α_p, so that α-regularity does not depend upon the coefficients a_p. If, in particular, $\alpha_1 = 1$, and (4.13) holds with $\omega = 0$, then the C-fraction is "absolutely regular" (Scott and Wall [4]). If $\alpha_p = 1$, $(p = 1, 2, 3, \cdots)$, then (4.13) holds with $\omega = 0$. In this case the C-fraction and its power series are called *semi-normal.*

Let us return for a moment to the blocks Q_p in the Padé table for a regular power series $P(z)$. The order r_p of Q_p expresses in a certain sense the amount of the degree of approximation of $A_p(z)/B_p(z)$ to $P(z)$ *in excess of normal.* It is therefore of interest to have upper and lower bounds for the numbers r_p. From (4.9) and (4.12) we readily obtain the inequalities

$$\begin{aligned}&\alpha_1+\alpha_3+\cdots+\alpha_{2p+1}-1\\&\qquad\geq r_{2p}\geq(\alpha_1+\alpha_3+\cdots+\alpha_{2p+1}-1)-(\alpha_2+\alpha_4+\cdots+\alpha_{2p}+\omega_p),\\&\alpha_2+\alpha_4+\cdots+\alpha_{2p+2}-1\\&\qquad\geq r_{2p+1}\geq(\alpha_2+\alpha_4+\cdots+\alpha_{2p+2}+\omega_p)-(\alpha_1+\alpha_3+\cdots+\alpha_{2p+1}).\end{aligned}$$

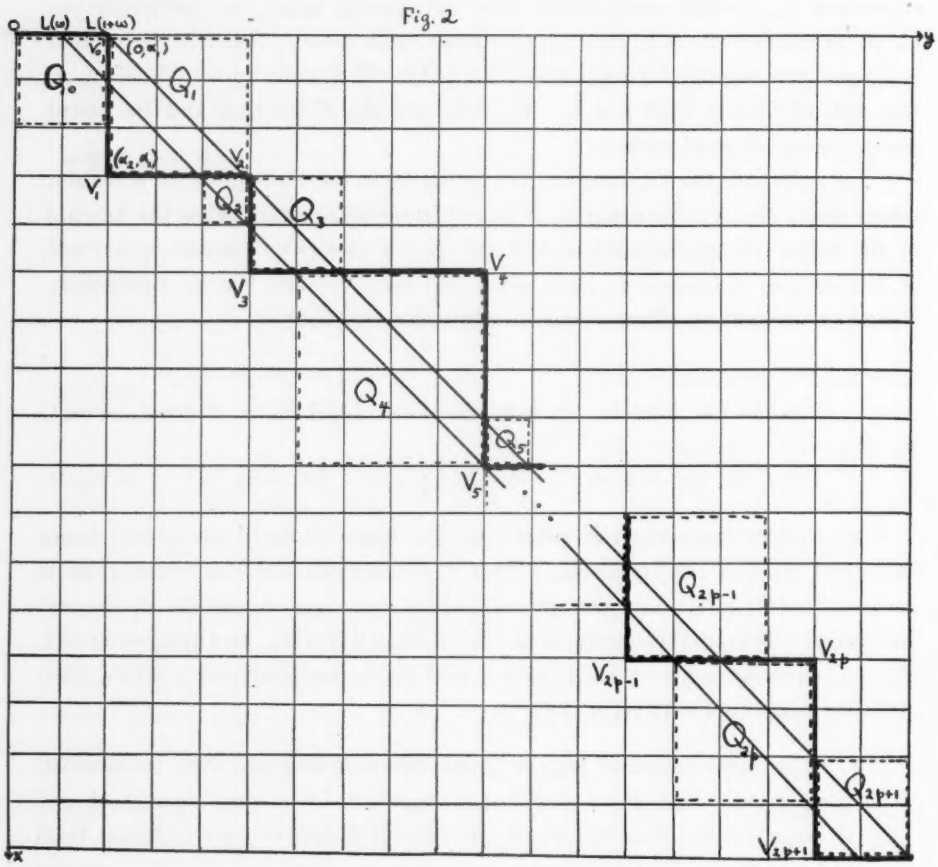
In certain cases, one can determine the exact values of the r_p and hence the exact sizes of the blocks Q_p. This depends upon whether equality holds in (4.12) (i), (ii). For example, if (a) $\alpha_1 = 1$, $\omega_p = 0$, and the a_p are real and positive or actual inequality holds in (4.12) (iii), (iv), or (b) $\omega_p = \alpha_1 - 1$, $\alpha_1 \neq 1$, equality holds in (4.12) (iv) and the a_p are real and positive, then equality holds in (4.12) (i), (ii).

5. Characterization of regular power series in terms of their coefficients. It is well known that $P(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is semi-normal if and only if certain determinants formed from the coefficients c_p are different from zero ([2], p. 304). If $P(z)$ is absolutely regular, Scott and Wall [4] showed that certain of these same determinants are different from zero, but they did not obtain the complete characterization. We shall now proceed to do this for the more general α-regular power series.

Let $P(z)$ be α-regular. Then, in the Padé table, the blocks $Q_0, Q_1, \cdots$ are all cut by the straight lines $L(\omega)$ and $L(1+\omega)$, for a suitable integer $\omega \geq 0$ (see Fig. 2). The line $L(\omega)$ passes through the file of approximants $f_{0,\omega}, f_{1,1+\omega}, f_{2,2+\omega}, \cdots$, and $L(1+\omega)$ passes through $f_{0,1+\omega}, f_{1,2+\omega}, f_{2,3+\omega}, \cdots$. All the even approximants of the C-fraction are contained in the first of these files while all the odd approximants are in the second. The approximants on

$L(\omega)$ which occupy squares in Q_{2p} are equal to one another and are different from those occupying squares outside Q_{2p}. Similarly, the approximants on $L(1+\omega)$ occupying squares in Q_{2p+1} are equal to one another, and are different from those outside Q_{2p+1}. Hence if we put

Fig. 2



$$(5.1)\quad g_0 = 0,\ g_p = \alpha_2 + \alpha_4 + \cdots + \alpha_{2p},\ h_p = \alpha_1 + \alpha_3 + \cdots + \alpha_{2p-1},\quad (p = 1, 2, 3, \cdots),$$

we have

$$f_{0,\omega} = f_{1,1+\omega} = \cdots = f_{h_1-\omega-1,h_1-1} \neq f_{h_1-\omega,h_1};$$
$$f_{h_p-\omega-2,h_p-1} \neq f_{h_p-\omega-1,h_p} = f_{h_p-\omega,h_p+1} = \cdots = f_{g_p-1,g_p+\omega} \neq f_{g_p,g_p+\omega+1};$$
$$f_{g_p-1,g_p+\omega-1} \neq f_{g_p,g_p+\omega} = f_{g_p+1,g_p+\omega+1} = \cdots = f_{h_{p+1}-\omega-1,h_{p+1}-1} \neq f_{h_{p+1}-\omega,h_{p+1}},$$
$$(p = 1, 2, 3, \cdots).$$

We may now apply Theorem 3.2 to obtain the following *necessary* conditions for $P(z)$ to be α-regular:

$$
\begin{aligned}
&\Delta_{i,\omega+1+i} = 0, \ (i = 0, 1, \cdots, h_1 - \omega - 2), \ \Delta_{h_1-\omega-1,h_1} \neq 0; \\
&\Delta_{h_p-\omega-2,h_p} \neq 0, \ \Delta_{h_p-\omega-1+i,h_p+1+i} = 0, \ (i = 0, 1, \cdots, g_p + \omega - h_p - 1), \\
&\Delta_{g_p-1,g_p+\omega+1} \neq 0 \\
&\Delta_{g_p-1,g_p+\omega} \neq 0, \ \Delta_{g_p+i,g_p+\omega+1+i} = 0, \ (i = 0, 1, \cdots, h_{p+1} - g_p - 2 - \omega), \\
&\Delta_{h_{p+1}} - 1 - \omega, h_{p+1} \neq 0, \ (p = 1, 2, 3, \cdots).
\end{aligned} \tag{5.2}
$$

If we change the notation by writing $\Phi(n, \omega) = \Delta_{n-1,n+\omega}$, $\Psi(n, \omega) = \Delta_{n-2,n+\omega}$, we obtain from (5.2) the conditions

$$
\begin{aligned}
&\Phi(g_p, \omega) \neq 0, \qquad \Phi(g_p + i, \omega) = 0, \ (i = 1, 2, \cdots, h_{p+1} - g_p - 1 - \omega), \\
&\Phi(h_{p+1} - \omega, \omega) \neq 0, \ (p = 0, 1, 2, \cdots); \\
&\Psi(h_p - \omega, \omega) \neq 0, \ \Psi(h_p + i - \omega, \omega) = 0, \ (i = 1, 2, \cdots, g_p + \omega - h_p), \\
&\Psi(g_p + 1, \omega) \neq 0, \ (p = 1, 2, 3, \cdots).
\end{aligned} \tag{5.3}
$$

We find by a process similar to that used by Scott and Wall [4] that the coefficients a_p in the C-fraction are given by the formulas

$$
\begin{aligned}
a_1 &= \frac{(-1)^{a_1-1-\omega}\Phi(h_1 - \omega, \omega)}{\Psi(h_1 - \omega, \omega)}, \qquad a_2 = \frac{(-1)^{a_2}\Psi(1 + g_1, \omega)}{\Phi(g_1, \omega)}, \\
a_{2p-1} &= \frac{(-1)^{a_{2p-1}}\Phi(h_p - \omega, \omega)\Psi(h_{p-1} - \omega, \omega)}{\Psi(h_p - \omega, \omega)\Phi(h_{p-1} - \omega, \omega)}, \\
a_{2p} &= \frac{(-1)^{a_{2p}}\Psi(1 + g_p, \omega)\Phi(g_{p-1}, \omega)}{\Psi(1 + g_{p-1}, \omega)\Phi(g_p, \omega)}, \qquad (p = 2, 3, 4, \cdots).
\end{aligned} \tag{5.4}
$$

These values of the a_p may also be obtained from the algorithm (2.2), (2.5).

We note that, by (4.13), the numbers g_p, h_p satisfy the inequalities

$$
\begin{aligned}
\omega + 1 \leq h_1 \leq g_1 + \omega \leq h_2 - 1 \\
\leq g_2 + \omega - 1 \leq h_2 - 2 \leq g_3 + \omega - 2 \leq \cdots.
\end{aligned} \tag{5.5}
$$

Moreover, if g_p and h_p are any numbers satisfying (5.5), then the C-fraction with exponents

$$
a_{2p} = g_p - g_{p-1}, \quad a_{2p-1} = h_p - h_{p-1}, \quad (g_0 = h_0 = 0), \tag{5.6}
$$

is α-regular.

We shall now prove that the conditions (5.3), which are necessary for α-regularity, are also *sufficient*. We suppose that (5.3) holds, where the g_p and h_p are any numbers satisfying (5.5), for some non-negative integer ω. We then determine numbers a_p by the formulas (5.4) and numbers a_p by (5.6), and form the C-fraction (1.1), which, by Theorem 4.2, is necessarily α-regular. This C-fraction has a power series expansion $P'(z) = 1 + c'_1 z + c'_2 z^2 + \cdots$. It is required to show that $c'_p = c_p$, $(p = 1, 2, 3, \cdots)$. From the Padé table for $P'(z)$ we see by what has already been proved that

(5.3) and (5.4) hold with the c_p replaced by the c'_p. It is then clear that the a_p may be determined from the algorithm of **2** if we start either with $P(z)$ or with $P'(z)$. Inasmuch as that algorithm determines the c_p uniquely in terms of the a_p for a given set of the α_p, it follows that $c'_p = c_p$, $(p = 1, 2, 3, \cdots)$.

We have proved the following theorem.

THEOREM 5.1. *A power series* $P(z) = 1 + c_1 z + c_2 z^2 + \cdots$ *is* α*-regular if and only if there exist positive integers* $\alpha_1, \alpha_2, \alpha_3, \cdots$ *and an integer* $\omega \geq 0$ *such that* (4.13) *holds and such that the coefficients* c_p *satisfy the conditions* (5.3), *where the* g_p *and* h_p *are defined by* (5.1).

As a corollary we have the well-known theorem that $P(z)$ is semi-normal if and only if the determinants $\Phi(p, 0)$, $\Psi(p, 0)$, $(p = 1, 2, 3, \cdots)$, are all different from zero.

Conditions for *regularity* similar to (5.2) can be set up in terms of the coefficients c_p, the parameters ω_p, and additional parameters ω'_p. The latter determine diagonal lines $L(\omega'_p)$ and $L(1 + \omega'_p)$ which intersect the common boundaries of Q_{2p+1} and Q_{2p+2}. Formulas analogous to (5.4) for the a_p may also be obtained. As these conditions and formulas are somewhat complicated, we shall omit them here.

6. Some transformations of *C*-fractions. Let $P(z) = 1 + c_1 z + c_2 z^2 + \cdots$ be a power series which does not represent a rational function of z, and let (1.1) be its C-fraction expansion. Let c_k be the first of the coefficients $c_1, c_2, c_3, \cdots$ which is not zero. Then the power series

$$(6.1) \quad P^*(z) = \frac{P(z) - 1}{c_k z^k}, \quad Q(z) = \frac{1}{P(z)}, \quad P_1(z) = P(z) + Cz^{a_1},$$

where C is a constant, do not represent rational functions of z, and must therefore have non-terminating C-fraction expansions. There are a number of cases where the coefficients and exponents in the C-fractions for the power series (6.1) can be expressed in a simple way in terms of the coefficients and exponents in the C-fraction for $P(z)$. The problem is clearly equivalent to the problem of finding the coefficients b_p, d_p, e_p and exponents β_p, δ_p, ϵ_p in the three continued fractions

$$(6.2) \quad 1 + a_1 z^{a_1} + \cfrac{b_1 z^{\beta_1}}{1} + \cfrac{b_2 z^{\beta_2}}{1} + \cfrac{b_3 z^{\beta_3}}{1} + \cdots,$$

$$(6.3) \quad \cfrac{1}{1} + \cfrac{d_1 z^{\delta_1}}{1} + \cfrac{d_2 z^{\delta_2}}{1} + \cfrac{d_3 z^{\delta_3}}{1} + \cdots,$$

and

$$1 - Cz^{\alpha_1} + \frac{e_1 z^{\epsilon_1}}{1} + \frac{e_2 z^{\epsilon_2}}{1} + \frac{e_3 z^{\epsilon_3}}{1} + \cdots, \tag{6.4}$$

which are all equal to (1.1) in the sense that the power series for (6.2), (6.3), (6.4), and (1.1) are all identical with one another. We shall obtain the required formulas in a number of cases.

THEOREM 6.1. *If the exponents α_p of (1.1) satisfy the inequalities*

$$\alpha_2 + \alpha_4 + \cdots + \alpha_{2p+2} > \alpha_3 + \alpha_5 + \cdots + \alpha_{2p+1} > \alpha_2 + \alpha_4 + \cdots + \alpha_{2p}, \quad (p = 1, 2, 3, \cdots), \tag{6.5}$$

then the coefficients b_p and exponents β_p in (6.2) are given by the formulas

$$\begin{aligned} &b_1 = -a_1 a_2,\ b_2 = a_2,\ \beta_1 = \alpha_1 + \alpha_2,\ \beta_2 = \alpha_2, \\ &b_{4p-1} = -b_{4p} = -\frac{a_3 a_5 \cdots a_{2p+1}}{a_2 a_4 \cdots a_{2p}},\quad b_{4p+1} = -b_{4p+2} = -\frac{a_2 a_4 \cdots a_{2p+2}}{a_3 a_5 \cdots a_{2p+1}}, \\ &\beta_{4p-1} = \beta_{4p} = (\alpha_3 + \alpha_5 + \cdots + \alpha_{2p+1}) - (\alpha_2 + \alpha_4 + \cdots + \alpha_{2p}), \\ &\beta_{4p+1} = \beta_{4p+2} = (\alpha_2 + \alpha_4 + \cdots + \alpha_{2p+2}) - (\alpha_3 + \alpha_5 + \cdots + \alpha_{2p+1}), \\ &\qquad (p = 1, 2, 3, \cdots). \end{aligned} \tag{6.6}$$

Proof. We take the even part of (6.2), i.e., we form the C-fraction whose approximants are the even approximants of (6.2) ([2], p. 201):

$$1 + a_1 z^{\alpha_1} + \frac{b_1 z^{\beta_1}}{1 + b_2 z^{\beta_2}} - \frac{b_2 b_3 z^{\beta_2+\beta_3}}{1 + b_3 z^{\beta_3} + b_4 z^{\beta_4}} - \frac{b_4 b_5 z^{\beta_4+\beta_5}}{1 + b_5 z^{\beta_5} + b_6 z^{\beta_6}} - \cdots. \tag{6.7}$$

We now determine the b_p and β_p by the formulas

$$\begin{aligned} &b_1 = -a_1 a_2,\quad b_2 = a_2,\ b_{2p} b_{2p+1} = -a_{p+2},\quad b_{2p+1} = -b_{2p+2}, \\ &\beta_1 = \alpha_1 + \alpha_2,\ \beta_2 = \alpha_2,\ \beta_{2p} + \beta_{2p+1} = \alpha_{p+2},\ \beta_{2p+1} = \beta_{2p+2}, \\ &\qquad (p = 1, 2, 3, \cdots), \end{aligned} \tag{6.8}$$

so that (6.7) becomes

$$1 + a_1 z^{\alpha_1} - \frac{a_1 a_2 z^{\alpha_1+\alpha_2}}{1 + a_2 z^{\alpha_2}} + \frac{a_3 z^{\alpha_3}}{1} + \frac{a_4 z^{\alpha_4}}{1} + \cdots. \tag{6.9}$$

By (6.5), the equations (6.8) can be solved for the β_p, and give (6.6). One may now readily verify that the approximants of (6.9) are the 1-st, 2-nd, 3-rd, $\cdots$ approximants of (1.1). It then follows immediately that the power series for (6.2), in which the b_p and β_p are given by (6.6), is the same as the power series for (1.1), and the theorem is proved.

Example. Theorem 6.1 may be applied to the C-fraction of Ramanujan ([3], p. 214), and gives the identity

$$(6.10)\quad 1+\frac{z}{1}+\frac{z^2}{1}+\frac{z^3}{1}+\frac{z^4}{1}+\cdots$$
$$=1+z-\frac{z^3}{1}+\frac{z^2}{1}-\frac{z}{1}+\frac{z}{1}-\frac{z^3}{1}+\frac{z^3}{1}-\frac{z^2}{1}+\frac{z^2}{1}-\frac{z^4}{1}+\frac{z^4}{1}-\cdots.$$

THEOREM 6.2. *If the exponents α_p of the C-fraction* (1.1) *satisfy the inequalities*

$$(6.11)\quad \alpha_1+\alpha_3+\cdots+\alpha_{2p+1}>\alpha_2+\alpha_4+\cdots+\alpha_{2p}>\alpha_1+\alpha_3+\cdots+\alpha_{2p-1},$$
$$(p=1,2,3,\cdots),$$

then the coefficients d_p and exponents δ_p in (6.3) *are given by the formulas*

$$d_1=-d_2=-a_1,\quad d_{4p-1}=-d_{4p}=-\frac{a_2a_4\cdots a_{2p}}{a_1a_3\cdots a_{2p-1}},$$
$$d_{4p+1}=-d_{4p+2}=-\frac{a_1a_3\cdots a_{2p+1}}{a_2a_4\cdots a_{2p}},$$
$$(6.12)\quad \delta_1=\delta_2=\alpha_1,\ \delta_{4p-1}=\delta_{4p}=(\alpha_2+\alpha_4+\cdots+\alpha_{2p})-(\alpha_1+\alpha_3+\cdots+\alpha_{2p-1}),$$
$$\delta_{4p+1}=\delta_{4p+2}=(\alpha_1+\alpha_3+\cdots+\alpha_{2p+1})-(\alpha_2+\alpha_4+\cdots+\alpha_{2p}),$$
$$(p=1,2,3,\cdots).$$

Proof. The odd part of (6.3), in which the d_p and δ_p are given by (6.12), is precisely (1.1). Hence (6.3) and (1.1) must have one and the same power series.

Example. If we apply Theorem 6.2 to the C-fraction of Ramanujan, we obtain the identity

$$(6.13)\quad 1+\frac{z}{1}+\frac{z^2}{1}+\frac{z^3}{1}+\cdots$$
$$=\frac{1}{1}-\frac{z}{1}+\frac{z}{1}-\frac{z}{1}+\frac{z}{1}-\frac{z^2}{1}+\frac{z^2}{1}-\frac{z^2}{1}+\frac{z^2}{1}-\frac{z^3}{1}+\frac{z^3}{1}-\frac{z^3}{1}+\frac{z^3}{1}-\cdots.$$

The approximants of (6.2) and (6.3) are of course rational approximants for $P(z)$, and are easily seen to be Padé approximants in case the power series $P^*(z)$ and $Q(z)$ of (6.1) are regular. Under the conditions of Theorems 6.1 and 6.2 it may be verified by means of Theorem 4.2 that $P^*(z)$ and $Q(z)$, respectively, are α-regular. Hence we have the following theorem:

THEOREM 6.3. *Under the conditions of Theorems* 6.1 *and* 6.2, *the approximants of* (6.2) *and* (6.3), *respectively, are all Padé approximants for $P(z)$.*

Example. In Fig. 3 are shown blocks in the Padé table for the Ramanujan C-fraction, which are occupied by the approximants of the latter and of the continued fractions in the right-hand members of (6.10) and (6.13). These blocks are designated by $0, 1, 2, \cdots$; $0^*, 1^*, 2^*, \cdots$; and $^*0, ^*1, ^*2, \cdots$,

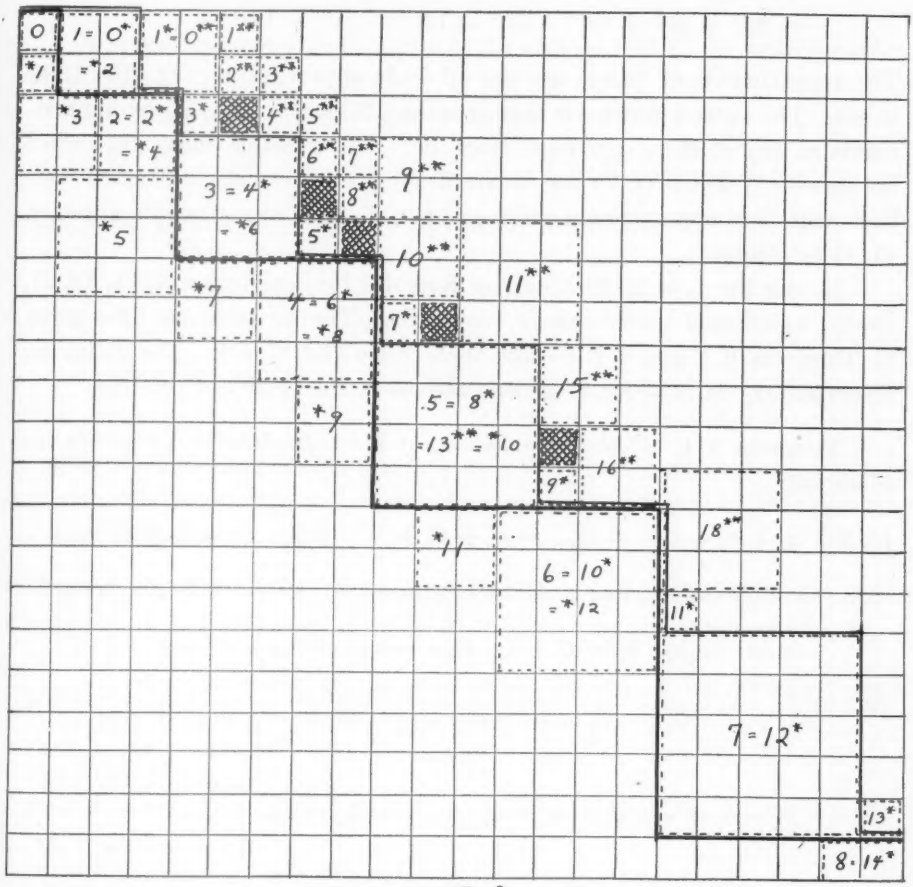


Fig. 3

respectively. The blocks designated by $0^{**}, 1^{**}, 2^{**}, \cdots$ are occupied by certain of the approximants of

$$(6.14)\quad 1 + z - z^3 + \cfrac{z^5}{1} \cfrac{z}{-1} + \cfrac{2z}{1} - \cfrac{3z/2}{1} \cfrac{z/6}{-1} - \cfrac{4z/3}{1} + \cfrac{3z}{1} \cfrac{\frac{1}{2}z}{-1} \cfrac{\frac{1}{2}z}{-1}$$
$$\cfrac{z}{-1} + \cfrac{2z^2}{1} \cfrac{z^3}{-1} + \cfrac{z}{1} \cfrac{z}{1} - \cfrac{z}{1} + \cfrac{z}{1} \cfrac{z^3}{-1} \cfrac{z}{-1} + \cfrac{z^3}{1} + \cdots,$$

which is also equal to the C-fraction of Ramanujan. This can be obtained by

means of the algorithm of **2**, if we start with the power series for the Ramanujan C-fraction, namely,

$$P(z) = 1 + z - z^3 + z^5 + z^6 - z^7 - 2z^8 + 2z^{10} + 2z^{11} - z^{12} - 3z^{13} - z^{14} + 3z^{15} + 3z^{16} - 2z^{17} - 5z^{18} - z^{19} + 6z^{20} + 5z^{21} - 3z^{22} - 8z^{23} - 2z^{24} + 8z^{25} + 7z^{26} - 5z^{27} - 12z^{28} - 2z^{29} + 13z^{30} + \cdots.$$

The approximants of (6.14) are not all Padé approximants, e. g., the 12-th is not. The squares which are shaded in Fig. 3 are not occupied by approximants of any of these continued fractions. This example shows that when the process by which (6. 2) was obtained is repeated, the new continued fractions may have approximants which are not all Padé approximants although (1. 1) is regular.

In case the $\alpha_p = 1$, Stieltjes gave formulas for computing (6. 2), (6. 3), (6. 4), which hold under certain conditions. The formulas we have given in Theorems 6. 1 and 6. 2 do not apply when the $\alpha_p = 1$. The following theorems may be regarded as generalizations of the results of Stieltjes.

THEOREM 6. 4. *If the exponents α_p of the C-fraction* (1. 1) *satisfy the condition*

$$(6.15)\quad \alpha_3 + \alpha_5 + \cdots + \alpha_{2p+1} = \alpha_2 + \alpha_4 + \cdots + \alpha_{2p}, \quad (p = 1, 2, 3, \cdots),$$

then the coefficients b_p and exponents β_p in (6. 2) *are given by the formulas*

$$b_1 = -a_1 a_2, \quad b_2 = a_2 + a_3, \quad b_{2p+1} = a_{2p+3} + a_{2p+2} - b_{2p+2},$$

$$(6.16)\quad b_{2p+2} = a_{2p+3} + \frac{a_{2p+2}}{1 + \frac{a_{2p+1}}{a_{2p}} + \frac{a_{2p+1}a_{2p-1}}{a_{2p}a_{2p-2}} + \cdots + \frac{a_{2p+1}a_{2p-1}\cdots a_3}{a_{2p}a_{2p-2}\cdots a_2}}, \quad (p = 1, 2, 3, \cdots);$$

$$\beta_1 = \alpha_1 + \alpha_2, \quad \beta_2 = \alpha_2 = \alpha_3, \quad \beta_{2p-1} = \beta_{2p} = \alpha_{2p} = \alpha_{2p+1}, \quad (p = 2, 3, 4, \cdots),$$

provided the values of the a_p are such that (6. 16) *can be solved for the $b_p \neq 0$.*

Theorem 6. 4 may be proved by comparing the odd part of (1. 1) with the even part of (6. 2).

THEOREM 6. 5. *If the exponents α_p of* (1. 1) *satisfy the condition*

$$(6.17)\quad \alpha_2 + \alpha_4 + \cdots + \alpha_{2p} = \alpha_1 + \alpha_3 + \cdots + \alpha_{2p-1}, \quad (p = 1, 2, 3, \cdots),$$

then the d_p and δ_p in (6. 3) *are given by the formulas*

$$
\begin{aligned}
&d_1 = -a_1, \quad d_2 = a_1 + a_2, \\
&d_{2p}d_{2p+1} = a_{2p}a_{2p+1}, \quad d_{2p+1} + d_{2p+2} = a_{2p+1} + a_{2p+2}, \\
&\delta_1 = \delta_2 = \alpha_1 = \alpha_2, \quad \delta_{2p+1} = \delta_{2p+2} = \alpha_{2p+1} = \alpha_{2p+2}, \quad (p = 1, 2, 3, \cdots),
\end{aligned} \tag{6.18}
$$

provided the a_p are such that (6.18) *can be solved for the $d_p \neq 0$.*

To prove Theorem 6.5, we compare the even part of (1.1) with the odd part of (6.3).

If we change the notation by substituting

$$a_1 = 1/k_1, \quad a_p = 1/k_{p-1}k_p, \quad d_1 = 1/f_1, \quad d_p = 1/f_{p-1}f_p, \quad (p = 2, 3, 4, \cdots),$$

we find that formulas (6.18) may be written in the form

$$
\begin{aligned}
&f_1 = -k_1, \quad f_2 = -k_2/(1 + k_2), \\
&f_{2p+1} = -k_{2p+1}(1 + k_2 + k_4 + \cdots + k_{2p})^2, \quad (p = 1, 2, 3, \cdots), \\
&f_{2p} = \frac{-k_{2p}}{(1 + k_2 + k_4 + \cdots + k_{2p-2})(1 + k_2 + k_4 + \cdots + k_{2p})}, \\
&\qquad\qquad (p = 2, 3, 4, \cdots).
\end{aligned} \tag{6.19}
$$

It must, of course, be assumed that $1 + k_2 + k_4 + \cdots + k_{2p} \neq 0$, $(p = 1, 2, 3, \cdots)$. These formulas are exactly the same as those found by Stieltjes in the case where the $\alpha_p = 1$.

THEOREM 6.6. *If the exponents α_p of* (1.1) *satisfy* (6.15), *then the coefficients e_p and exponents ϵ_p of* (6.4) *are given by the formulas*

$$
\begin{aligned}
&e_1 = -a_1a_2, \; e_2 = a_2 + a_3, \; e_{2p}e_{2p+1} = a_{2p+1}a_{2p+2}, \; e_{2p+1} + e_{2p+2} = a_{2p+2} + a_{2p+3}, \\
&\epsilon_1 = \alpha_1 + \alpha_2, \; \epsilon_2 = \alpha_2 = \alpha_3, \; \epsilon_{2p+1} = \epsilon_{2p+2} = \alpha_{2p+2} = \alpha_{2p+3}, \\
&\qquad\qquad (p = 1, 2, 3, \cdots),
\end{aligned} \tag{6.20}
$$

provided $a_1 = -C$. If $a_1 \neq -C$, the formulas relating the a_p, α_p, e_p, and ϵ_p are

$$
\begin{aligned}
&e_1 = a_1 + C, \quad e_{2p-1}e_{2p} = a_{2p-1}a_{2p}, \quad e_{2p} + e_{2p+1} = a_{2p} + a_{2p+1}, \\
&\epsilon_1 = \alpha_1, \quad \epsilon_{2p} = \epsilon_{2p+1} = \alpha_{2p} = \alpha_{2p+1}, \qquad (p = 1, 2, 3, \cdots).
\end{aligned} \tag{6.21}
$$

The values of the a_p must be such that conditions (6.20) *and* (6.21) *are solvable for the $e_p \neq 0$.*

Proof. The even part of (6.4), in which the e_p and ϵ_p are given by (6.20), is the same as the odd part of (1.1). Therefore (6.4) and (1.1) must have the same power series. Similarly, the odd part of (6.4), in which the e_p and ϵ_p are given by (6.21), is precisely the odd part of (1.1). Again the power series for (6.4) and (1.1) must be the same.

Formulas (6.21) are equivalent to those found by Stieltjes for the case where the $\alpha_p = 1$. For, if we let

$$a_1 = 1/k_1, \quad a_p = 1/k_{p-1}k_p, \quad e_1 = 1/h_1, \quad e_p = 1/h_{p-1}h_p, \quad (p = 2, 3, 4, \cdots),$$

then we find that formulas (6.21) become

$$(6.22)\quad \begin{aligned} h_1 &= k_1/(1 + Ck_1), \quad h_{2p} = k_{2p}(1 + C(k_1 + k_3 + \cdots + k_{2p-1}))^2, \\ h_{2p+1} &= \frac{k_{2p+1}}{(1 + C(k_1 + k_3 + \cdots + k_{2p-1}))(1 + C(k_1 + k_3 + \cdots + k_{2p+1}))}, \\ &\qquad (p = 1, 2, 3, \cdots). \end{aligned}$$

One may readily verify the following theorem by means of Theorem 4.2.

THEOREM 6.7. *Under the conditions of Theorems* 6.4, 6.5, *and* 6.6, *the approximants of* (6.2), (6.3), *and* (6.4), *respectively, are all Padé approximants for* $P(z)$.

By analogous devices, one can obtain the continued fractions (6.2), (6.3), and (6.4) in other cases, e. g., where the α_p satisfy (6.11) for certain values of p and (6.15) for the remaining values. To do this, one must form, from the odd and even parts of (1.1), continued fractions which have all but a finite number of the approximants of (1.1), as, for instance, the continued fraction (6.9).

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THE BERNSTEIN POLYNOMIALS FOR DISCONTINUOUS FUNCTIONS.*

By FRITZ HERZOG and J. D. HILL.

1. Introduction. A classical theorem of Weierstrass asserts that every continuous function defined on a finite closed interval can be uniformly approximated by polynomials. There are several well known proofs of this theorem in the literature; among these is the very elegant demonstration of S. Bernstein,[1] who showed that if $f(x)$ is continuous in $[0,1]$[2] then the sequence of polynomials

$$B_n(f;x) \equiv \sum_{k=0}^{n} f(k/n)T_{nk}(x) \qquad (n=1,2,3,\cdots), \tag{1.1}$$

where

$$T_{nk}(x) \equiv C_{n,k}x^k(1-x)^{n-k}, \tag{1.2}$$

converges uniformly to $f(x)$ in $[0,1]$.

For later reference it is convenient at this point to observe the following obvious facts.

$$\begin{aligned} &a \leqq f(k/n) \leqq A \text{ for all } k/n \text{ in } [0,1] \text{ implies} \\ &\quad a \leqq B_n(f;x) \leqq A \text{ for all } x \text{ in } [0,1]. \end{aligned} \tag{1.3}$$

$$B_n(f;0) = f(0); \qquad B_n(f;1) = f(1). \tag{1.4}$$

The elegance of Bernstein's theorem lies in the facts that the approximating polynomials are given explicitly and that they depend only on the values of the function $f(x)$ for rational values of x. This latter fact leads us to introduce at once the following definitions.

(1.5) *Definition.* By a *skeleton* we shall mean a function $f(r)$ defined only for rational values of r in $[0,1]$.

On the other hand, in this paper we shall hereafter reserve the term *function* to denote an $f(x)$ defined for all x in $[0,1]$.

(1.6) *Definition.* By the *class* $\mathfrak{S}$ we shall mean the class of all skeletons $f(r)$ for which the limits

* Received September 4, 1945.

[1] S. Bernstein, *Communications de la Société Mathématique de Kharkov*, (2), vol. 13 (1912), pp. 1-2.

[2] By $[a,b]$ or (a,b) we mean the interval $a \leqq x \leqq b$ or $a < x < b$, respectively.

$$f_L(x) \equiv \lim_{r \to x-} f(r) \qquad (0 < x \leqq 1), \tag{1.7}$$

$$f_R(x) \equiv \lim_{r \to x+} f(r) \qquad (0 \leqq x < 1), \tag{1.8}$$

exist for the values of x indicated.

(1.9) *Definition.* For each $f \in \mathfrak{S}$ we shall understand by the *normalized extension* of $f(r)$ the function $f_N(x)$ defined by the use of (1.7) and (1.8) as follows.

$$f_N(x) \equiv \tfrac{1}{2}[f_L(x) + f_R(x)] \qquad (0 < x < 1), \tag{1.10}$$

$$f_N(0) \equiv f(0); \qquad f_N(1) \equiv f(1).^3 \tag{1.11}$$

(1.12) *Definition.* Two arbitrary skeletons $f(r)$ and $g(r)$ are called *equivalent*, and we write $f(r) \sim g(r)$, if for each $\epsilon > 0$ the inequality $|f(r) - g(r)| \geqq \epsilon$ holds for at most a finite number of values of r.

We are now able to state precisely the object of this paper. We propose to investigate the behavior of the polynomials (1.1), which we shall call the *Bernstein polynomials* of f, for the case of a bounded skeleton f.

In **2** we collect in the form of lemmas certain results which are used throughout. In **3** we establish some useful lemmas concerning skeletons of the class $\mathfrak{S}$ and their normalized extensions.

The behavior of the Bernstein polynomials of a skeleton f, under the single restriction that f be bounded, is discussed in **4**. Here we obtain as a principal result Theorem (4.3), which gives lower and upper bounds, respectively, for $\underline{\lim}_n B_n(f;x)$ and $\overline{\lim}_n B_n(f;x)$ for each x. In **5** we derive the following results concerning skeletons f of the class $\mathfrak{S}$. In the first place we show that the Bernstein polynomials of such a skeleton converge in $[0,1]$ to its normalized extension (Theorem (5.1)); and that this convergence is uniform in any closed subinterval in which the normalized extension is continuous (Theorem (5.5)). In the second place we show that if f and g are two skeletons in $\mathfrak{S}$ then $B_n(f;x)$ and $B_n(g;x)$ converge to the same limit function if and only if f and g are equivalent (Theorem (5.6)).

In **6** we derive sufficient conditions for the convergence of the Bernstein polynomials of an arbitrary skeleton f in terms of the order of magnitude of the sum $\sum_{k=0}^{n} |f(k/n) - g(k/n)|$, where g is a skeleton in $\mathfrak{S}$. A necessary condition for convergence in terms of the order of magnitude of $\sum_{k=0}^{n} [f(k/n) - g(k/n)]$ is also given.

---

[3] This somewhat unnatural definition at the endpoints is justified by its convenience See (1.4).

Finally in 7 we present some examples to illustrate the main features of the theory.

2. Preliminary lemmas. The majority of our results depend upon one or more of the following lemmas. For the definition of $T_{nk}(x)$ we refer the reader to (1.2).

(2.1) LEMMA. *For $n = 1, 2, 3, \cdots$; $k = 0, 1, 2, \cdots, n$ and $0 < x < 1$, we have $T_{nk}(x) < K[nx(1-x)]^{-\frac{1}{2}}$, where K is an absolute constant.*

Proof. We shall show that $n^{\frac{1}{2}}C_{n,k}x^{k+\frac{1}{2}}(1-x)^{n-k+\frac{1}{2}}$ is bounded for all values of n, k and x. As a function of x this expression assumes its maximum value for $x = (k+\frac{1}{2})/(n+1)$. This maximum value may be written in the form

$$\frac{n^{\frac{1}{2}}n!}{(n+1)^{n+1}} \cdot \frac{(k+\frac{1}{2})^{k+\frac{1}{2}}}{k!} \cdot \frac{(n-k+\frac{1}{2})^{n-k+\frac{1}{2}}}{(n-k)!}. \tag{2.2}$$

The second and third factors in (2.2) have the form $A_m = (m+\frac{1}{2})^{m+\frac{1}{2}}/m!$. By the use of Stirling's formula we find that $A_m < K_1 e^m$ for $m \geqq 0$, and that the first factor in (2.2) is less than $K_2 e^{-n}$, where K_1 and K_2 are certain constants. Combining these estimates we find that the expression in (2.2) is less than a constant K.

(2.3) LEMMA. *For $n = 1, 2, 3, \cdots$ and $0 \leqq x \leqq 1$ we have $\sum_k^* T_{nk}(x) < 1/(4n^{\frac{1}{2}})$, where the sum is taken over all k for which $|k - nx| > n^{3/4}$.*

This is a known result.[4]

(2.4) LEMMA. *For $n = 1, 2, 3, \cdots$ and $0 < x < 1$ we have*

$$\lim_{n\to\infty} \sum_{0\leqq k<nx} T_{nk}(x) = \tfrac{1}{2} = \lim_{n\to\infty} \sum_{nx<k\leqq n} T_{nk}(x).$$

This follows at once from the central limit theorem of probability, namely, that

$$\lim_{n\to\infty} \sum_k T_{nk}(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{t} e^{-u^2/2}du,$$

where the sum is taken over those values of k for which $0 \leqq k < nx + t[nx(1-x)]^{\frac{1}{2}}$. Merely set $t = 0$.

(2.5) LEMMA. *For $n = 1, 2, 3, \cdots$ and $0 < x < 1$ we have*

$$\lim_{n\to\infty} \sum_k{}' T_{nk}(x) = \tfrac{1}{2} = \lim_{n\to\infty} \sum_k{}'' T_{nk}(x),$$

[4] G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 1, Berlin (1925), p. 66, no. 145.

where Σ_k' *is taken over all* k *for which* $nx - n^{3/4} \leqq k < nx$ *and* Σ_k'' *over all* k *for which* $nx < k \leqq nx + n^{3/4}$.

This is a direct consequence of Lemmas (2.3) and (2.4).

(2.6) LEMMA. *If* $F(x)$ *is bounded in* $[0,1]$ *and continuous at a point* x_0 *in* $(0,1)$ *then the sequence* $\{B_n(F;x_0)\}$ *converges to* $F(x_0)$. *Moreover if* $F(x)$ *is continuous at every point*[5] *of* $[a,b]$ *where* $0 \leqq a < b \leqq 1$, *then the Bernstein polynomials* $B_n(F;x)$ *converge uniformly to* $F(x)$ *in* $[a,b]$.

For the case in which $a = 0$ and $b = 1$ this is precisely Bernstein's original theorem (see **1**). In the general case appropriate modifications of the proof given by Pólya and Szegö[6] suffice to establish the lemma as stated.

3. The class $\mathfrak{S}$. In (1.5) we introduced the concept of a skeleton and in (1.6) we defined the class $\mathfrak{S}$ of skeletons. We now proceed to establish several results concerning skeletons of the class $\mathfrak{S}$. The first of these is the following lemma.

(3.1) LEMMA. *If* $f(r) \in \mathfrak{S}$ *then* $f_L(x)$ *and* $f_R(x)$ *defined by* (1.7) *and* (1.8) *are at most simply discontinuous functions for* $0 < x \leqq 1$ *and* $0 \leqq x < 1$, *respectively, and*

$$f_L(x-) = f_R(x-) = f_L(x) \qquad (0 < x \leqq 1), \tag{3.2}$$

$$f_L(x+) = f_R(x+) = f_R(x) \qquad (0 \leqq x < 1). \tag{3.3}$$

Proof. Let x_0 be a fixed point in the interval $0 < x \leqq 1$. Equation (3.2) is equivalent to the assertion that for an arbitrary sequence $\{x_n\}$ converging to x_0 from the left, we have

$$\lim_{n\to\infty} f_L(x_n) = \lim_{n\to\infty} f_R(x_n) = f_L(x_0). \tag{3.4}$$

To establish (3.4) we associate with each x_n two rational numbers r_n' and r_n'', such that $x_n - 1/n < r_n' < x_n < r_n'' < x_0$ and such that

$$|f(r_n') - f_L(x_n)| < 1/n; \quad |f(r_n'') - f_R(x_n)| < 1/n. \tag{3.5}$$

We now make use of the following identities.

$$f_L(x_n) = f_L(x_0) + [f(r_n') - f_L(x_0)] - [f(r_n') - f_L(x_n)], \tag{3.6}$$

$$f_R(x_n) = f_L(x_0) + [f(r_n'') - f_L(x_0)] - [f(r_n'') - f_R(x_n)]. \tag{3.7}$$

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[5] By this we mean that if $a > 0$ then $F(x)$ is to be continuous on the left at a as well as on the right; an analogous statement holds with respect to b if $b < 1$.

[6] G. Pólya and G. Szegö, *loc. cit.*, p. 66, no. 146.

Since $r_n' \to x_0 -$ and $r_n'' \to x_0 -$ the differences in the first brackets on the right sides of (3.6) and (3.7) approach zero, while the second brackets do so because of (3.5). Consequently (3.6) and (3.7) imply (3.4). In a similar fashion we may establish (3.3).

The next lemma, which is a simple consequence of the preceding, involves the notion of the normalized extension (see (1.9)) of a skeleton $f \epsilon \mathfrak{S}$.

(3.8) LEMMA. *The normalized extension $f_N(x)$ of a skeleton $f(r) \epsilon \mathfrak{S}$ is at most a simply discontinuous function in $[0, 1]$. Moreover, the following relations hold.*

$$f_N(x-) = f_L(x) \qquad (0 < x \leqq 1), \tag{3.9}$$

$$f_N(x+) = f_R(x) \qquad (0 \leqq x < 1), \tag{3.10}$$

$$f_N(x) = \tfrac{1}{2}[f_N(x-) + f_N(x+)] \qquad (0 < x < 1). \tag{3.11}$$

Proof. Let x_0 be any point in the interval $0 < x \leqq 1$. By Lemma (3.1) the limits $f_L(x_0 -)$ and $f_R(x_0 -)$ exist, so that $f_N(x_0 -)$ exists by (1.10). Equation (3.9) then follows at once from (1.10) and (3.2). Similarly, for an x_0 in the interval $0 \leqq x < 1$, $f_N(x_0 +)$ exists and (3.10) holds. Equation (3.11) now follows directly from (1.10), (3.9) and (3.10).

For a skeleton $f(r) \epsilon \mathfrak{S}$ the values of $f_N(r)$ and $f(r)$ are not necessarily equal. For example, if $f(r) \equiv 1/m$ when $r = h/m$ in its lowest terms, then $f_N(x) = 0$ in $(0, 1)$, and thus $f_N(r)$ is actually different from $f(r)$ for all values of r in $(0, 1)$. The following lemma, however, establishes the fact that $f(r)$ and $f_N(r)$ are always equivalent in the sense of (1.12).

(3.12) LEMMA. *If $f(r)$ is a skeleton of the class $\mathfrak{S}$ and if $f_N(x)$ is its normalized extension, then $f_N(r) \sim f(r)$.*

Proof. According to Definition (1.12) we have to show that for each $\epsilon > 0$ the inequality $|f_N(r) - f(r)| \geqq \epsilon$ is satisfied by at most a finite number of values of r. Suppose, on the contrary, that for a certain $\epsilon_0 > 0$ the inequality $|f_N(r) - f(r)| \geqq \epsilon_0$ holds for infinitely many r. Among these values of r there is a sequence $\{r_n\}$ such that either $r_n \to x_0 -$ or $r_n \to x_0 +$, where x_0 is some point in $[0, 1]$. We assume for definiteness that $r_n \to x_0 -$ so that $0 < x_0 \leqq 1$. By (3.9) we have $\lim_n f_N(r_n) = f_L(x_0)$, and by (1.7) we have $\lim_n f(r_n) = f_L(x_0)$. Consequently we find that $\lim_n [f_N(r_n) - f(r_n)] = 0$, which contradicts the assumption that $|f_N(r_n) - f(r_n)| \geqq \epsilon_0$ for all n. A similar argument holds if $r_n \to x_0 +$.

The connection between equivalent skeletons in $\mathfrak{S}$ and their normalized extensions is shown by the following result.

(3.13) LEMMA. *Two skeletons in $\mathfrak{S}$ are equivalent if and only if their normalized extensions are identical in* $(0,1)$.

Proof. Let $f(r)$ and $g(r)$ be two skeletons in $\mathfrak{S}$ and let $f_N(x)$ and $g_N(x)$ be their normalized extensions. Then if we set $h(r) = f(r) - g(r)$ it follows from (1.10) that $h_N(x) = f_N(x) - g_N(x)$ in $(0,1)$. Now if $h_N(x) = 0$ in $(0,1)$ then $h(r) \sim 0$ by Lemma (3.12) and hence $f(r) \sim g(r)$. Conversely, if $f(r) \sim g(r)$ then $h(r) \sim 0$, from which it follows at once that $h_N(x) = 0$ in $(0,1)$.

4. The Bernstein polynomials for bounded skeletons. For any skeleton $f(r)$, bounded or not, we now define the following four auxiliary functions.

$$(4.1) \qquad \underline{f}_L(x) \equiv \underline{\lim_{r\to x-}} f(r); \quad \overline{f}_L(x) \equiv \overline{\lim_{r\to x-}} f(r) \qquad (0 < x \leqq 1),$$

$$(4.2) \qquad \underline{f}_R(x) \equiv \underline{\lim_{r\to x+}} f(r); \quad \overline{f}_R(x) \equiv \overline{\lim_{r\to x+}} f(r) \qquad (0 \leqq x < 1).$$

It can be easily shown that the finiteness of these four limits for the values of x indicated is sufficient as well as necessary for the boundedness of $f(r)$.

We recall that the Bernstein polynomials $B_n(f;x)$ for an arbitrary skeleton $f(r)$ were defined in (1.1). We come now to our first main result which gives the best possible restriction on the behavior of the sequence $\{B_n(f;x)\}$ in $(0,1)$, as f ranges over the class of bounded skeletons. On account of (1.4) there is no convergence problem for $x = 0$ or 1.

(4.3) THEOREM. *For an arbitrary bounded skeleton $f(r)$ the following inequalities hold for $0 < x < 1$.*

$$(4.4) \qquad \underline{\lim_{n\to\infty}} B_n(f;x) \geqq \tfrac{1}{2}[\underline{f}_L(x) + \underline{f}_R(x)],$$

$$(4.5) \qquad \overline{\lim_{n\to\infty}} B_n(f;x) \leqq \tfrac{1}{2}[\overline{f}_L(x) + \overline{f}_R(x)].$$

Proof. We shall give the proof of (4.5). The proof of (4.4) will then follow by applying (4.5) to $-f$. Let x_0 be a fixed point in $(0,1)$. We shall write (1.1) in the form

$$(4.6) \qquad B_n(f;x_0) = \sum_k{}' f(k/n)T_{nk}(x_0) + \sum_k{}'' f(k/n)T_{nk}(x_0) + \sum_k{}^* f(k/n)T_{nk}(x_0) + X,$$

where here and in what follows the symbols $\sum_k{}'$, $\sum_k{}''$, $\sum_k{}^*$ denote summation

over those values of k for which $nx_0 - n^{3/4} \leqq k < nx_0$, $nx_0 < k \leqq nx_0 + n^{3/4}$, $|k - nx_0| > n^{3/4}$, respectively. The term X corresponds to $k = nx_0$ in case nx_0 is an integer and is equal to 0 otherwise. In either case by Lemma (2.1) we have $X = o(1)$ as $n \to \infty$.

Now let λ_n' and λ_n'' denote the least upper bounds of $f(r)$ for $x_0 - n^{-1/4} \leqq r < x_0$ and $x_0 < r \leqq x_0 + n^{-1/4}$, respectively. By (4.1) and (4.2) we have

$$\lim_{n\to\infty} \lambda_n' = \bar{f}_L(x_0); \quad \lim_{n\to\infty} \lambda_n'' = \bar{f}_R(x_0). \tag{4.7}$$

Furthermore, from the definitions of $\sum_k'$ and $\sum_k''$, we have

$$\sum_k' f(k/n) T_{nk}(x_0) \leqq \lambda_n' \sum_k' T_{nk}(x_0), \tag{4.8}$$

$$\sum_k'' f(k/n) T_{nk}(x_0) \leqq \lambda_n'' \sum_k'' T_{nk}(x_0). \tag{4.9}$$

By (4.7) and Lemma (2.5) the right sides of (4.8) and (4.9) approach $\frac{1}{2}\bar{f}_L(x_0)$ and $\frac{1}{2}\bar{f}_R(x_0)$, respectively, as $n \to \infty$. Finally, if M is an upper bound for $f(r)$ then

$$\sum_k^* f(k/n) T_{nk}(x_0) \leqq M \sum_k^* T_{nk}(x_0). \tag{4.10}$$

By Lemma (2.3) the right side of (4.10) tends to 0 as $n \to \infty$. The preceding estimates taken in conjunction with (4.6) clearly establish (4.5).

(4.11) COROLLARY. *For an arbitrary bounded skeleton $f(r)$ the following inequalities*[7] *hold for $0 < x < 1$.*

$$\underline{\lim_{r\to x}} f(r) \leqq \underline{\lim_{n\to\infty}} B_n(f;x) \leqq \overline{\lim_{n\to\infty}} B_n(f;x) \leqq \overline{\lim_{r\to x}} f(r).$$

In **7** we shall give examples (see (7.1) and (7.3)) which show that for a bounded skeleton $f(r)$ no more than (4.4) and (4.5) can be said in general. Indeed, by a combination of examples of the types (7.1) and (7.3) it is possible to establish the following fact: If x_0 is any point in $(0,1)$ then there exists a skeleton $f(r)$ for which the six quantities $\underline{f}_L(x_0)$, $\bar{f}_L(x_0)$, $\underline{f}_R(x_0)$, $\bar{f}_R(x_0)$, $\underline{\lim}_n B_n(f;x_0)$ and $\overline{\lim}_n B_n(f;x_0)$ assume any six given values a', A', a'', A'', λ and Λ, respectively, for which $\frac{1}{2}(a' + a'') \leqq \lambda \leqq \Lambda \leqq \frac{1}{2}(A' + A'')$, $a' \leqq A'$ and $a'' \leqq A''$.

5. The Bernstein polynomials for skeletons in $\mathfrak{S}$. In this section we shall be concerned for the most part with skeletons f of the class $\mathfrak{S}$. There

[7] By "$\underline{\lim_{r\to x}} f(r)$" we mean the limit as δ approaches 0 of the greatest lower bound of $f(r)$ for $0 < |r - x| < \delta$; and analogously for the upper limit.

are several questions that naturally arise in connection with such skeletons. The first of these is the following: Does the sequence $\{B_n(f;x)\}$ converge and, if so, what is the limit function? This question is completely answered in the next theorem.

(5.1) THEOREM. *If $f(r)$ is a skeleton of class $\mathfrak{S}$ the sequence of its Bernstein polynomials converges for $0 \leqq x \leqq 1$ to $f_N(x)$, the normalized extension of $f(r)$.*

Proof. By (4.1) and (1.7), $\underline{f}_L(x) = \overline{f}_L(x) = f_L(x)$ and, by (4.2) and (1.8), $\underline{f}_R(x) = \overline{f}_R(x) = f_R(x)$ for $0 < x < 1$. Hence the right sides of (4.4) and (4.5) are equal and their common value, by (1.10), is $f_N(x)$. Thus from Theorem (4.3) we conclude that $\lim_n B_n(f;x)$ exists for $0 < x < 1$ and equals $f_N(x)$. At $x = 0$ and $x = 1$ the result is clear from (1.4) and (1.11).

It is of interest to observe in passing that the proof of the preceding theorem serves to establish the following result, which is local in character and does not require that f belong to $\mathfrak{S}$. If $f(r)$ is a bounded skeleton and if at a point x_0 in $(0,1)$ the limits in (1.7) and (1.8) exist then we have $\lim_n B_n(f;x_0) = \frac{1}{2}[f_L(x_0) + f_R(x_0)]$.

Before investigating the question of when the convergence assured by Theorem (5.1) is uniform, we find it convenient to establish the following lemma.

(5.2) LEMMA. *Let $g(r)$ be a skeleton satisfying the following conditions:* (i) *$g(r)$ is bounded;* (ii) *$g_R(0) = g(0) = 0$ and $g_L(1) = g(1) = 0$;* (iii) *for each fixed $\epsilon > 0$, $\sum^{(1)}_k |g(k/n)| = o(n^{\frac{1}{2}})$ as $n \to \infty$, where the symbol $\sum^{(1)}_k$ denotes summation over those values of k for which $|g(k/n)| \geqq \epsilon$. Then the sequence $\{B_n(g;x)\}$ converges uniformly to zero in $[0,1]$.*

Proof. Let $\epsilon > 0$ be fixed arbitrarily. By condition (ii) there exists a δ $(0 < \delta < 1/4)$ such that $|g(r)| < \epsilon$ for $0 \leqq r \leqq 2\delta$ and for $1 - 2\delta \leqq r \leqq 1$. For each $n = 1, 2, 3, \cdots$ we write the Bernstein polynomial of $g(r)$ in the form

$$B_n(g;x) = \sum^{(1)}_k g(k/n) T_{nk}(x) + \sum^{(2)}_k g(k/n) T_{nk}(x), \tag{5.3}$$

where $\sum^{(1)}_k$ has already been defined in (iii) above and $\sum^{(2)}_k$ is extended to those values of k for which $|g(k/n)| < \epsilon$. It is obvious that the second term on

the right in (5.3) does not exceed ϵ in absolute value for $0 \leqq x \leqq 1$ and $n = 1, 2, 3, \cdots$. In order to estimate the first term on the right in (5.3) we consider first the case in which $\delta \leqq x \leqq 1 - \delta$. In this case, by applying Lemma (2.1), we obtain

$$|\sum_k^{(1)} g(k/n)T_{nk}(x)| < K[nx(1-x)]^{-\frac{1}{2}} \sum_k^{(1)} |g(k/n)|.$$

By condition (iii) it is clear that the right-hand side approaches 0 uniformly in $[\delta, 1-\delta]$ as $n \to \infty$. On the other hand, for x in $[0, \delta]$ or $[1-\delta, 1]$ we have

$$|\sum_k^{(1)} g(k/n)T_{nk}(x)| \leqq M \sum_k^{(1)} T_{nk}(x),$$

where M is abound on the absolute value of $g(r)$. For the values of k involved in $\sum_k^{(1)}$ and for the values of x under consideration it is not difficult to see that $|k/n - x| > \delta$. Hence for all n such that $n^{-1/4} < \delta$ we have $\sum_k^{(1)} T_{nk}(x) \leqq \sum_k^{*} T_{nk}(x)$, where $\sum_k^{*}$ is taken over those values of k for which $|k - nx| > n^{3/4}$. Consequently, by applying Lemma (2.3), we obtain

$$|\sum_k^{(1)} g(k/n)T_{nk}(x)| < M/(4n^{\frac{1}{2}}),$$

which is valid for all values of n sufficiently large and for all values of x in $[0, \delta]$ and $[1-\delta, 1]$. The proof is therefore complete.

(5.4) COROLLARY. *If the skeleton $g(r)$ is equivalent to zero and if $g(0) = g(1) = 0$ then the Bernstein polynomials of g approach zero uniformly in $[0, 1]$.*

We return now to the following question. Under what conditions do the Bernstein polynomials of a skeleton in $\mathfrak{S}$ converge uniformly? A complete answer to this question is given in our next theorem.

(5.5) THEOREM. *If $f(r)$ is a skeleton in the class $\mathfrak{S}$ then the Bernstein polynomials of $f(r)$ converge uniformly in $[a, b]$ where $0 \leqq a < b \leqq 1$, if and only if the limit function $f_N(x)$ is continuous in $[a, b]$.*

Proof. The necessity of the condition is trivial. Assume then that $f_N(x)$ is continuous in $[a, b]$. We first make the following remark concerning the continuity of $f_N(x)$ at the endpoints of this interval. (In this connection

see footnote [5].) If $a > 0$ then our assumption merely states that $f_N(a+) = f_N(a)$. However, by (3.11) we conclude at once that $f_N(a-) = f_N(a)$, so that $f_N(x)$ is continuous at $x = a$ from the left as well as from the right. An analogous statement holds for $x = b$ in case $b < 1$. Thus in any case we are able to apply Lemma (2.6) and obtain the result that $B_n(f_N; x)$ converges uniformly to $f_N(x)$ in $[a, b]$. Hence in view of the identity

$$B_n(f; x) - f_N(x) = [B_n(f; x) - B_n(f_N; x)] + [B_n(f_N; x) - f_N(x)],$$

the proof will be complete if we can show that $B_n(f; x) - B_n(f_N; x)$ converges to 0 uniformly in $[a, b]$. This will be accomplished by showing that the preceding difference converges to 0 uniformly in the whole interval $[0, 1]$.

Let $g(r) \equiv f(r) - f_N(r)$ for rational r in $[0, 1]$. From (1.1) it is clear that $B_n(g; x) = B_n(f; x) - B_n(f_N; x)$. By Lemma (3.12) we have $f_N(r) \sim f(r)$ and hence $g(r) \sim 0$. Finally, by (1.11) we have $g(0) = g(1) = 0$. We therefore conclude from Corollary (5.4) that $B_n(g; x)$ converges to 0 uniformly in $[0, 1]$. This completes the proof.

We come now to the final question of this section. Under what conditions will the Bernstein polynomials corresponding to two skeletons in $\mathfrak{S}$ converge to the same limit function? This question is answered in the following theorem in which, on account of (1.4), we restrict ourselves to the open interval $(0, 1)$.

(5.6) THEOREM. *If $f(r)$ and $g(r)$ are two skeletons in $\mathfrak{S}$ then $\lim_n B_n(f; x) = \lim_n B_n(g; x)$ for $0 < x < 1$ if and only if $f(r) \sim g(r)$.*

Proof. The proof is immediate. By Theorem (5.1) the limit functions in question are $f_N(x)$ and $g_N(x)$. By Lemma (3.13) $f_N(x) = g_N(x)$ for $0 < x < 1$ if and only if $f(r) \sim g(r)$.

6. Further convergence theorems. It is natural to ask at this point if any interesting convergence theorems for $\{B_n(f; x)\}$ can be obtained by relaxing the restriction that f belong to $\mathfrak{S}$. The Examples (7.1) and (7.3) in **7** below will make it plain that in general no such theorems are to be expected if we stray "too far" from $\mathfrak{S}$. On the other hand, if we stay "close enough" to $\mathfrak{S}$, in the sense of (6.2), (6.6) and (6.9) below, some further results concerning the convergence of $\{B_n(f; x)\}$ can be deduced.

We shall begin with the following result which establishes a necessary condition for convergence.

(6.1) THEOREM. *In order that the Bernstein polynomials of a given bounded skeleton $f(r)$ converge everywhere in $(0, 1)$, or even almost every-*

where, to a function $g(x)$, which is at most simply discontinuous in $[0,1]$, it is necessary that

$$\sum_{k=0}^{n}[f(k/n)-g(k/n)]=o(n) \qquad (n\to\infty). \tag{6.2}$$

Proof. Since the points of discontinuity of $g(x)$ form at most a denumerable set it is clear from the first part of Lemma (2.6) that $B_n(g;x)$ converges to $g(x)$ almost everywhere in $(0,1)$. Hence $B_n(f-g;x)$ approaches 0 almost everywhere in $(0,1)$. Since by (1.3) the sequence $\{B_n(f-g;x)\}$ is uniformly bounded in $[0,1]$, we conclude from the Lebesgue convergence theorem that $\int_0^1 B_n(f-g;x)dx=o(1)$. The condition (6.2) now follows from the well known formula

$$\int_0^1 T_{nk}(x)dx=1/(n+1) \quad (k=0,1,\cdots,n;\ n=1,2,3,\cdots). \tag{6.3}$$

(6.4) COROLLARY. *In order that the Bernstein polynomials of a given bounded skeleton $f(r)$ converge to zero almost everywhere in $(0,1)$ it is necessary that $\sum_k f(k/n)=o(n)$ as $n\to\infty$.*

If we strengthen (6.2) by taking the absolute values of the differences $f(k/n)-g(k/n)$ we obtain the sufficient condition (6.6) for the convergence almost everywhere of a subsequence of $\{B_n(f;x)\}$. The comparative weakness of this conclusion is perhaps to be expected since if $f(r)$ is bounded then the left-hand side of (6.6) is certainly $O(n)$.

(6.5) THEOREM. *If for a given skeleton $f(r)$ there exists a skeleton $g(r)$ in $\mathfrak{S}$ such that*

$$\sum_{k=0}^{n}|f(k/n)-g(k/n)|=o(n) \qquad (n\to\infty), \tag{6.6}$$

then there is a subsequence of $\{B_n(f;x)\}$ which converges to $g_N(x)$ almost everywhere in $(0,1)$.

Proof. From Theorem (5.1) it follows that the sequence $\{B_n(g;x)\}$ converges everywhere in $(0,1)$ to $g_N(x)$. Hence it will suffice to show the existence of a sequence of indices $\{n_i\}$ such that $\lim_i B_{n_i}(f-g;x)=0$ almost everywhere. To do this we start with the simple inequality

$$|B_n(f-g;x)|\leqq\sum_{k=0}^{n}|f(k/n)-g(k/n)|\,T_{nk}(x). \tag{6.7}$$

This relation and the formula (6.3) immediately yield the inequality

$$\int_0^1 |B_n(f-g;x)|\,dx \leqq [1/(n+1)] \sum_{k=0}^{n} |f(k/n)-g(k/n)|.$$

From the hypothesis (6.6) it follows that the integral on the left is $o(1)$ as $n \to \infty$. Hence by a familiar result [8] there exists a sequence of indices $\{n_i\}$ such that $B_{n_i}(f-g;x)$ approaches 0 almost everywhere in $(0,1)$ as $i \to \infty$. This completes the proof.

Finally, if we strengthen (6.6) to $o(n^{\frac{1}{2}})$ we can prove the convergence of $\{B_n(f;x)\}$ for $0 < x < 1$.

(6.8) THEOREM. *If for a given skeleton $f(r)$ there exists a skeleton $g(r)$ in $\mathfrak{S}$ such that*

$$\sum_{k=0}^{n} |f(k/n)-g(k/n)| = o(n^{\frac{1}{2}}) \qquad (n \to \infty), \tag{6.9}$$

then the sequence $\{B_n(f;x)\}$ converges to $g_N(x)$ for $0 < x < 1$.

Proof. Following the method of the preceding proof we have merely to show that the sequence $\{B_n(f-g;x)\}$ converges to 0 for $0 < x < 1$. But this follows at once from (6.9) by applying Lemma (2.1) to the $T_{nk}(x)$ in (6.7).

(6.10) COROLLARY. *If for a given skeleton $f(r)$ we have $\sum_k |f(k/n)| = o(n^{\frac{1}{2}})$, then $\lim_n B_n(f;x) = 0$ for $0 < x < 1$.*

There is a considerable "gap" between the necessary condition (6.2) and the sufficient condition (6.9). However, by means of Examples (7.5) and (7.8), given in the next section, we establish the fact that this gap cannot be narrowed. Consequently, it is impossible to give a necessary *and* sufficient condition for convergence in terms of the sums involved. Furthermore, by means of examples, which we refrain from including, we have been able to show that in condition (6.2) of Theorem (6.1) the differences on the left side cannot be replaced by their absolute values, while in condition (6.9) of Theorem (6.8) the absolute values of the differences on the left side cannot be replaced by the differences themselves.

7. Examples. This section is devoted to a number of examples which illustrate certain aspects of the theory developed in **4-6**. In this connection we call attention to the remarks made at the end of **4** and at the end of **6**.

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[8] E. W. Hobson, *The Theory of Functions of a Real Variable*, vol. 2, Cambridge (1926), p. 245.

Throughout this section we shall use the generic notation $r = h/m$, where h and m are relatively prime non-negative integers, to represent rational numbers r in $[0, 1]$ in their lowest terms. Without further emphasis we shall frequently employ the simple fact that $f(r) \equiv f(h/m)$ will appear among the $f(k/n)$ for $k = 0, 1, \cdots, n$ if and only if m is a divisor of n.

(7.1) *Example.* The four quantities occurring in (4.4) and (4.5) may be related in the following manner for $0 < x < 1$,

$$(7.2)\quad \tfrac{1}{2}[\underline{f}_L(x) + \underline{f}_R(x)] = \underline{\lim_{n\to\infty}} B_n(f; x) < \overline{\lim_{n\to\infty}} B_n(f; x) = \tfrac{1}{2}[\overline{f}_L(x) + \overline{f}_R(x)].$$

Let x_0 be a fixed number in $(0, 1)$ and let a' and a'' be two arbitrary positive numbers. If $r = h/m$ with m odd we define $f(r)$ as a' for r in $(0, x_0)$ and as a'' for r in $(x_0, 1)$. Otherwise we define $f(r)$ as 0. It is seen at once that $\underline{f}_L(x) = \underline{f}_R(x) = 0$ for $0 < x < 1$. Moreover, it is clear that (i) $\overline{f}_L(x) = \overline{f}_R(x) = a'$ for $0 < x < x_0$, (ii) $\overline{f}_L(x) = \overline{f}_R(x) = a''$ for $x_0 < x < 1$, and (iii) $\overline{f}_L(x_0) = a'$ and $\overline{f}_R(x_0) = a''$. Now if n is any power of 2 then $B_n(f; x) = 0$ for $0 < x < 1$. On the other hand, we conclude from Lemmas (2.3) and (2.4) that as $n \to \infty$ through odd values, $B_n(f; x)$ approaches (i) a' if $0 < x < x_0$, (ii) a'' if $x_0 < x < 1$, and (iii) $\frac{1}{2}(a' + a'')$ if $x = x_0$. Hence the relation (7.2) follows from Theorem (4.3).

The preceding example illustrates a fact which is not surprising, namely, that the Bernstein polynomials for a badly behaved skeleton may themselves behave very badly. The following example, however, shows that the Bernstein polynomials of a badly behaved skeleton may behave quite well.

(7.3) *Example.* The four quantities occurring in (4.4) and (4.5) may be related in the following manner for $0 < x < 1$,

$$\tfrac{1}{2}[\underline{f}_L(x) + \underline{f}_R(x)] < \underline{\lim_{n\to\infty}} B_n(f; x) = \overline{\lim_{n\to\infty}} B_n(f; x) < \tfrac{1}{2}[\overline{f}_L(x) + \overline{f}_R(x)],$$

and the convergence implied may even be uniform in $[0, 1]$.

This example depends on the construction of a sequence of rational numbers r_i of the form $h_i/2^i$ with h_i odd ($i = 1, 2, 3, \cdots$), such that each of the subsequences $\{r_{2j}\}$ and $\{r_{2j-1}\}$ is everywhere dense in $(0, 1)$. We leave it to the reader to convince himself that such a construction is possible. We now define the skeleton $f(r)$ as follows: $f(r_i) = (-1)^i r_i(1 - r_i)$ for all i and $f(r) = 0$ otherwise. It is obvious that $\underline{f}_L(x) = \underline{f}_R(x) = -x(1-x) < 0$ and $\overline{f}_L(x) = \overline{f}_R(x) = x(1-x) > 0$ for $0 < x < 1$. We now appeal to Lemma (5.2). Since conditions (i) and (ii) are clearly satisfied it remains

only to consider condition (iii). It will suffice to show that $\sum_{k=0}^{n} |f(k/n)| = O(\log n)$. But this is immediate since the number of non-zero terms in the preceding sum is equal to the exponent of the highest power of 2 that divides n. This exponent is plainly $O(\log n)$. Hence $\lim_n B_n(f;x) = 0$ uniformly in $[0,1]$.

Most of the theory developed in **5** can be illustrated by the following classical example from elementary analysis.

(7.4) *Example.* Let $f(r)$ be defined as follows: $f(0) = f(1) = 0$; when $0 < r < 1$ and $r = h/m$, $f(r) = 1/m$ or, more generally, $f(r) = a_m$ where $\{a_m\}$ is an arbitrary null sequence.

It is at once clear that $f_N(x) = 0$ for $0 \leqq x \leqq 1$ and hence, by Theorems (5.1) and (5.5), $B_n(f;x)$ approaches 0 uniformly in that interval.

The following counter-example will make it clear that in condition (6.2) of Theorem (6.1) the right side cannot be replaced by anything of order smaller than $o(n)$.

(7.5) *Example.* Let c_n be an arbitrary null sequence of non-negative numbers. Then it is possible to find a skeleton $f(r)$ for which (i) $\lim_n B_n(f;x) = 0$ in $(0,1)$, and (ii) $\sum_k f(k/n) \geqq c_n n$ for $n = 2, 3, 4, \cdots$. (See Corollary (6.4).)

In fact we may use for $f(r)$ the skeleton of Example (7.4) with the a_m equal to $2 \operatorname{Max}_{i \geqq m} c_i$. Statement (i) follows immediately from the fact that a_m tends to 0. To verify the truth of (ii) we first observe that if h/m is the reduced form of k/n $(k = 1, 2, \cdots, n-1)$ then $f(k/n) = a_m$. Moreover, since the a_m are monotonically non-increasing we conclude that $f(k/n) \geqq a_n$. Hence we have $\sum_k f(k/n) \geqq (n-1)a_n \geqq 2(n-1)c_n \geqq c_n n$ for $n \geqq 2$.

As our next example we propose to show that in condition (6.9) of Theorem (6.8) the right side cannot be replaced by anything of order larger than $o(n^{\frac{1}{2}})$. Before proceeding with the details of the example itself we first make the following remarks. (a) By evident modifications in the proof [4] of Lemma (2.3) it can be shown that for $0 \leqq x \leqq 1$

$$\Sigma^{\mathrm{I}} T_{nk}(x) > 3/4; \quad \Sigma^{\mathrm{II}} T_{nk}(x) < 1/4, \tag{7.6}$$

where the sums Σ^{I} and Σ^{II} are taken over those values of k for which $|k - nx| \leqq n^{\frac{1}{2}}$ and $|k - nx| > n^{\frac{1}{2}}$, respectively. (b) For any finite set $a_1, a_2, \cdots, a_s$ of real numbers, with each $a_i \geqq 2$, the following inequality can be easily established by induction.

$$a_1 + a_2 + \cdots + a_s \leqq a_1 a_2 \cdots a_s. \tag{7.7}$$

(7.8) *Example.* Let x_0 be any point in $(0,1)$. Then it is possible to find a skeleton $f(r)$ for which (i) $B_n(f;x_0)$ does not approach 0, and (ii) $\sum_k |f(k/n)| = O(n^{\frac{1}{2}})$. (See Corollary (6.10).)

It is clear first of all that there exists a number $P \geqq 5$ such that the interval $I_q \equiv [x_0 - q^{-\frac{1}{2}}, x_0 + q^{-\frac{1}{2}}]$ lies entirely in $(0,1)$ for all $q \geqq P$. For $r = h/m$ we now define $f(r)$ as 1 if m is a prime $p \geqq P$ and $r \epsilon I_p$. We define $f(r)$ as 0 otherwise. From this definition, from the meaning of Σ^{I}, and from (7.6) it follows that $B_p(f;x_0) = \Sigma^{\mathrm{I}} T_{nk}(x_0) > 3/4$ for all primes $p \geqq P$. This establishes (i). In order to show that (ii) holds, we make the preliminary observation that for any prime $p \geqq P$ the sum $\sum_k f(k/p)$ contains as many non-vanishing terms as there are integers in the interval $[px_0 - p^{\frac{1}{2}}, px_0 + p^{\frac{1}{2}}]$. Since the number of such integers is less than $2p^{\frac{1}{2}} + 1$, we have $\sum_k f(k/p) < 2p^{\frac{1}{2}} + 1 < 3p^{\frac{1}{2}}$. Now let n be an arbitrary positive integer. If n has no prime factors $\geqq P$ then $\sum_k f(k/n) = 0$. In the contrary case let $p_1, p_2, \cdots, p_s$ be the distinct primes $\geqq P$ which divide n. We may then write the sum $\sum_k f(k/n)$ in the form $\sum_{i=1}^{s} \sum_j f(j/p_i)$ which, by the remark made above, is less than $3 \sum_i p_i^{\frac{1}{2}}$. But the latter by (7.7) does not exceed $3 \prod_i p_i^{\frac{1}{2}}$ since $p_i^{\frac{1}{2}} \geqq P^{\frac{1}{2}} > 2$. We conclude therefore that $\sum_k f(k/n) < 3n^{\frac{1}{2}}$ for all values of n, which implies (ii).

The preceding example is, of course, open to the criticism that it establishes the non-convergence of $\{B_n(f;x)\}$ only at a single point x_0. It is possible, however, to modify the definition of $f(r)$ in such a way that $\{B_n(f;x)\}$ fails to approach 0 for any value of x in $(0,1)$, while (ii) remains satisfied. Since the details of this modification are moderately complicated we have contented ourselves with the simpler example above.

(7.9) *Example.* We define a skeleton $f(r)$ as follows. Let $f(r) = 1$ for r in (a,b) where $0 < a < b < 1$, and let $f(r) = 0$ otherwise. It is evident that f belongs to $\mathfrak{S}$ and that for its normalized extension we have (a) $f_N(x) = 0$ for $0 \leqq x < a$ and for $b < x \leqq 1$, (b) $f_N(x) = 1$ for $a < x < b$, and (c) $f_N(a) = f_N(b) = \frac{1}{2}$. By Theorem (5.1) the sequence $\{B_n(f;x)\}$ converges to $f_N(x)$ for all x in $[0,1]$. Furthermore, by Theorem (5.5) this convergence is uniform in any closed subinterval of $[0,1]$ which contains neither a nor b.

Let us now assign to x a fixed value p and set $q = 1 - p$. Then $B_n(f;p)$ assumes the form

$$\sum_{na < k < nb} C_{n,k} p^k q^{n-k}. \tag{7.10}$$

In view of the preceding discussion we see that the sum (7.10) approaches the limit (i) 0 if $0 \leqq p < a$ or $b < p \leqq 1$, (ii) 1 if $a < p < b$, (iii) $\frac{1}{2}$ if $p = a$ or $p = b$. If we interpret p as the constant probability of an event E then the sum (7.10) is precisely the probability that the relative frequency of the occurrence of E in n trials lies between a and b. Consequently, the results (i)-(iii) for the value of the limit of the sum (7.10) have evident interpretations in probability. In particular, if (a, b) is regarded as an arbitrary neighborhood of p then the result (ii) is exactly the weak law of large numbers. (As a matter of fact this law can also be deduced from the first part of Lemma (2.6) the proof of which does not involve the central limit theorem.) It is therefore conceivable that the theory developed in this paper may find other applications to questions in probability.

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ASYMPTOTIC EQUILIBRIA.*

By AUREL WINTNER.

1. A class of standard problems concerning the asymptotic behavior of solutions of linear differential equations centers around a simple "Abelian" theorem, which may be formulated as follows (cf. [3]): If (f_{ik}), where $i, k = 1, \cdots, n$, is a matrix of continuous functions $f_{ik} = f_{ik}(t)$, $0 \leqq t < \infty$, satisfying the n^2 conditions

$$\int^{\infty} |f_{ik}(t)|\, dt < \infty, \tag{1}$$

then each of the n components $x_i(t)$ of every solution of the linear differential equations

$$x_i' = \sum_{k=1}^{n} f_{ik}(t) x_k \tag{2}$$

tends to a finite limit as $t \to \infty$. In view of the primitive nature of this Abelian fact, it can be expected that some corresponding result holds in the general case of non-linear differential equations

$$x_i' = f_i(t; x_1, \cdots, x_n), \tag{3}$$

where $f_1, \cdots, f_n$ are continuous functions of position on the $(n+1)$-dimensional region

$$0 \leqq t < \infty; \qquad -\infty < x_1 < \infty, \cdots, -\infty < x_n < \infty \tag{4}$$

and satisfy an additional restriction which, corresponding to (1), compels the existence of n finite limits $x_i(\infty)$ for every solution $x_i(t)$ of (3). In other words, *every* solution *path* $x_i = x_i(t)$ should tend, as $t \to \infty$, to a certain *point* of the x-space, a point of asymptotic equilibrium.

The relevant Abelian theorem delimiting this situation will turn out to be as follows:

(i) *Let n real-valued, continuous functions f_i of the position $(t; x_1, \cdots, x_n)$ on the $(n+1)$-dimensional region* (4) *be such that*

$$\int^{\infty} |f_i(t; 0, \cdots, 0)|\, dt < \infty \tag{5}$$

* Received May 26, 1945.

and that there exists a matrix (λ_{ik}) *of* n^2 *non-negative, measurable functions*

$$\lambda_{ik} = \lambda_{ik}(t), \qquad 0 \leqq t < \infty \tag{6}$$

satisfying

$$\int^{\infty} \lambda_{ik}(t)\,dt < \infty \tag{7}$$

and

$$|f_i(t; x_1^*, \cdots, x_n^*) - f_i(t; x_1^{**}, \cdots, x_n^{**})| \leqq \sum_{k=1}^{n} \lambda_{ik}(t)\,|x_k^* - x_k^{**}|, \tag{8}$$

where t, x_k^*, x_k^{**} *are arbitrary. Suppose, in addition, that the functions* $\lambda_{ik}(t)$ *are bounded for* $0 \leqq t < T$ *if* $T < \infty$ *(but not necessarily if* $T = \infty$*). Then each of the n components* $x_i(t)$ *of a solution of the differential equations* (3) *tends to a finite limit,* $x_i(\infty)$, *as* $t \to \infty$, *no matter what the n initial values* $x_i(0)$ *determining the solution may be.*

Thus, $(x_1(0), \cdots, x_n(0)) \to (x_1(\infty), \cdots, x_n(\infty))$ is a mapping defined on the whole x-space. It is a question of stability whether the inverse of this mapping exists on the *whole* x-space and is single-valued and continuous. In this connection, cf. [1].

It may be mentioned that, in virtue of (8) and (7), condition (5) is equivalent to

$$\int^{\infty} |f_i(t; x_1^0, \cdots, x_n^0)|\,dt < \infty, \tag{9}$$

where $(x_1^0, \cdots, x_n^0)$ is an arbitrary point of the x-space. In order to see this, it is sufficient to choose $x_k^* = x_k^0$, $x_k^{**} = 0$.

2. The integral (5) is 0 if

$$f_i(t; x_1, \cdots, x_k) = \sum_{k=1}^{n} f_{ik}(t)\,x_k, \tag{10}$$

that is, if (3) is the linear system (2). It is also clear that condition (8) then is satisfied by the continuous functions $\lambda_{ik}(t) = |f_{ik}(t)|$; so that (7) becomes precisely (1). Hence, the fact quoted after (1) is contained in (i).

Correspondingly, an application made in [3] of the linear case, (2), of (i) can now be extended to non-linear differential equations, as follows:

(ii) *Let a matrix* (c_{ik}) *of* n^2 *real constants* $c_{11}, \cdots, c_{nn}$ *be such as to have linear elementary divisors and purely imaginary characteristic numbers only (so that the general solution of the differential equations*

$$y_i' = \sum_{k=1}^{n} c_{ik} y_k \tag{11}$$

is represented by n almost-periodic polynomials

$$y_i(t) = \sum_{k=1}^{n} \alpha_{ik} \cos(\mu_k t + \beta_{ik}), \tag{12}$$

where $-|\mu_1|^2, \cdots, -|\mu_n|^2$ *are the squares of the characteristic numbers of* (c_{ik}) *and* α_{ik}, β_{ik} *denote* $2n^2$ *integration constants, only n of which are independent). Then, if* $f_1, \cdots, f_n$ *are functions satisfying the assumptions of* (i), *there belongs to every solution* $(x_1(t), \cdots, x_n(t))$ *of the differential equations*

$$x_i' = \sum_{k=1}^{n} c_{ik} x_k + f_i(t; x_1, \cdots, x_n) \tag{13}$$

a solution (12) *of the trivial approximation,* (11), *to* (13) *in such a way that*

$$x_i(t) - y_i(t) \to 0 \text{ as } t \to \infty, \tag{14}$$

where $i = 1, \cdots, n$.

Since 0 is a purely imaginary number, the algebraic assumptions of (ii) are satisfied if (c_{ik}) is the zero matrix. In this case, every component $y_i(t)$ of every solution (12) of (11) is constant. Hence, the assertion, (14), of (ii) becomes the assertion of (i). Since (13) becomes (3), it follows that (ii) is a generalization of (i).

Actually, the more elaborate theorem, (ii), is equivalent to the purely Abelian lemma, (i). In fact, (ii) can be reduced to (i) by the same device of "variation of constants" which, in [3], reduced the linear case of (ii) to the linear case of (i). Inasmuch as not even the details of this reduction need a rewording, it will be sufficient to prove (i).

3. Even if the functions f_i are of class C' (or, for that matter, regular-analytic) on the whole region (4), a solution $x_i(t)$ of (3) which is determined by a set of n initial values $x_i(0)$ will not, in general. exist except for small t. In fact, the general existence theorem of non-linear differential equations is purely local in nature (even in the regular-analytic case). Hence, *the unrestricted existence of every solution* $x_i(t)$ *of* (3), (13) *throughout the whole range* $0 \leqq t < \infty$ *is part of the statements of* (i), (ii).

Correspondingly, the proof of (i) will fall into two parts. That part dealing with the unrestricted existence of the solutions will be taken care of by (iv) below. The remaining part, that concerning the existence of the limits $x_i(\infty)$, can then be formulated in a manner sharper than what is claimed by (i); namely, as follows:

(iii) *Let* $f_1, \cdots, f_n$ *satisfy the assumptions of* (i) *except its additional*

assumption made after (8); *so that the functions* $\lambda_{ik}(t)$ *need not now be bounded on bounded t-intervals. The remaining assumptions of* (i) *imply the existence of a fixed value T having the following property: Any solution* $(x_1(t), \cdots, x_n(t))$ *of* (3) *which exists at* $t = T$ *(for instance, any solution of* (3) *satisfying the initial conditions*

$$x_i(T) = x_i^0, \tag{15}$$

where $x_1^0, \cdots, x_n^0$ *are arbitrary) can be prolonged for every t past T, and all n components* $x_i(t)$, $T \leqq t < \infty$, *of the resulting solutions of* (3) *will tend to finite limits* $x_i(\infty)$.

That assumption of (i) which is omitted in (iii) means the existence of a Lipschitz constant $L = L_T$ satisfying

$$|f_i(t; x_1^*, \cdots, x_n^*) - f_i(t; x_1^{**}, \cdots, x_n^{**})| \leqq L \sum_{k=1}^{n} |x_k^* - x_k^{**}| \tag{16}$$

for unspecified x_k^*, x_k^{**} if $0 \leqq t \leqq T$, where T is arbitrarily fixed ($< \infty$). Hence, in order to reduce (i) to (iii), it is sufficient to choose T so large as required by (iii) and then to apply, on the resulting ranges $T \leqq t < \infty$ and $0 \leqq t \leqq T$, the respective assertions of (iii) and of the following lemma:

(iv) *Let n real-valued, continuous functions of position on an* $(n+1)$*-dimensional region*

$$0 \leqq t \leqq T; \qquad -\infty < x_1 < \infty, \cdots, -\infty < x_n < \infty \tag{17}$$

be such as to satisfy a uniform Lipschitz condition (16), *where* $0 \leqq t \leqq T$ *(and* x_k^*, x_k^{**} *are arbitrary). Then the solution* $x_i(t)$ *of* (3) *determined by any set of initial values* $x_i(0)$ *exists not only for "sufficiently small" t but on the whole t-range,* $0 \leqq t \leqq T$, *admitted in* (17).

This lemma, which is undoubtedly well-known, is of independent interest, since it assures existence in the large for all solutions. It may be verified, for instance, as follows: An application of (16) to $x_k^* = x_k$, $x_k^{**} = 0$ gives

$$|f_i(t; x_1, \cdots, x_n)| \leqq L \sum_{k=1}^{n} |x_k| + |f_i(t; 0, \cdots, 0)|,$$

where L is constant on the region (17). Since $f_i(t; 0, \cdots, 0)$ is a continuous, hence bounded, function on the interval $0 \leqq t \leqq T$, it follows that there exists a constant satisfying

$$|f_i(t; x_1, \cdots, x_n)| \leqq \text{const. max}\,(|x_1|, \cdots, |x_n|, 1)$$

on the region (17). Hence, (iv) is contained in the criteria proved in [2].

4. Since (iii) and (iv) imply (i), and since (ii) is equivalent to (i), only (iii) remains to be proved.

In the assumptions of (iii), the functions $\lambda_{ik}(t)$ occurring in (8) can be unbounded on every interval $t_0 < t < t_0 + \epsilon$ (in fact, they can be ∞ on a dense t-set, and remain ∞ on a dense t-set after *any* alteration of their values on t-sets of measure 0), since only the L-integrability of the functions $\lambda_{ik}(t)$ is required. In particular, (iii) does not assume any local Lipschitz condition (16) (for bounded t-ranges). Nevertheless, (iii) can be proved by the method of successive approximations. This will be shown by an adaptation of that "accruing" variant of the usual estimate of successive approximations on which Lichtenstein's fundamental existence theorems for the movement of gravitating fluid masses are based (conditions (8), (7) there correspond to such data as a Hölder condition and the behavior of potential or velocity fields at infinity).

According to (7),

$$\sum_{k=1}^{n} \int_{T}^{\infty} \lambda_{ik}(t)\,dt \leqq \theta < 1 \qquad (i = 1, \cdots, n), \tag{18}$$

if $T = T_\theta$ is large enough. It will be shown that every T satisfying (18) for some $\theta = \theta_T < 1$ is a T having the property claimed by (iii).

For fixed θ and T, let t be restricted to the interval $T \leqq t < \infty$. Then all solutions of (3) can be thought of as belonging to initial conditions (15), where the n constants x_i^0 are arbitrary. Let these initial values be fixed. According to (15), the corresponding successive approximations are defined by the recursion formula

$$x_i^{m+1}(t) = x_i^0 + \int_{T}^{t} f_i(u; x_1^m(u), \cdots, x_n^m(u))\,du, \tag{19}$$

where $m = 0, 1, \cdots$ and

$$x_i^0(t) = x_i^0 \text{ for all } t \tag{20}$$

$(i = 1, \cdots, n)$. This defines continuous functions $x_1^m(t), \cdots, x_n^m(t)$ for every t and for every m, since the functions $f_i(t; x_1, \cdots, x_n)$ are defined and continuous on the whole range (4).

Let C_m denote the least common upper bound of the n functions

$$|x_1^{m+1}(t) - x_1^m(t)|, \cdots, |x_n^{m+1}(t) - x_n^m(t)| \tag{21}$$

on the interval $T \leqq t < \infty$ (so that, for the present, $C_m \leqq \infty$). Since (20) and the case $m = 0$ of (19) imply that

$$|x_i^1(t) - x_i^0(t)| \leqq \int_T^t |f_i(u; x_1^0, \cdots, x^0)|\, du,$$

the formulation (9) of the assumption (5) of (iii) assures that $C_1 < \infty$. On the other hand, if m in (19) is replaced by $m-1$, it follows by subtraction that

$$|x_i^{m+1}(t) - x_i^m(t)| \leqq \int_T^t \sum_{k=1}^n \lambda_{ik}(u)\, |x_k^m(u) - x_k^{m-1}(u)|\, du,$$

by the case $x_k^* = x_k^m(u)$, $x_k^{**} = x_k^{m-1}(u)$ of (8). But the definition of C_m shows that the last integral is majorized by

$$\int_T^t \sum_{k=1}^n \lambda_{ik}(u) C_{m-1} du \leqq C_{m-1} \sum_{k=1}^n \int_T^\infty \lambda_{ik}(u) du,$$

a value which, according to (18), does not exceed $C_{m-1}\theta$. Hence, if the definition of C_m is compared with the last two formula lines, it follows that $C_m \leqq C_{m-1}\theta$. In view of $C_1 < \infty$, this means that there exists a constant $C_0 = C_0(\theta)$ satisfying $C_m \leqq C_0\theta^m$ for every m.

Since the constant C_m majorizes the n functions (21) for $T \leqq t < \infty$, the inequality $C_m \leqq C_0\theta^m$, where $\theta < 1$ by assumption, implies that the n series

$$\sum_{m=0}^\infty \{x_i^{m+1}(t) - x_i^m(t)\}$$

are uniformly convergent on the interval $T \leqq t < \infty$ and represent there n bounded functions. It follows therefore from the constancy of the functions (20) that, as $m \to \infty$, each of the n functions $x_i^m(t)$ tends to a limit function uniformly for $T \leqq t < \infty$ and that, if $x_i(t)$ denotes the limit function, $x_i(t)$ remains bounded as $t \to \infty$.

In particular, uniform convergence takes place on every bounded portion of the range $T \leqq t < \infty$. This in itself implies that $x_i(t)$ represents a solution of (3) for $T \leqq t < \infty$. Hence, the proof of (iii) will be complete if it is shown that the n limits $x_i(\infty)$ exist for the solution just constructed.

To this end, choose $x_k^* = x_k(t)$ and $x_k^{**} = x_k^0$ in (8). This gives

$$|f_i(t; x_1(t), \cdots, x_n(t))| \leqq |f_i(t; x_1^0, \cdots, x_n^0)| + \sum_{k=1}^n \lambda_{ik}(t)\, |x_k(t) - x_k^0|.$$

But the functions $x_1(t), \cdots, x_n(t)$ were seen to be bounded for $T \leqq t < \infty$. Hence, the function multiplying $\lambda_{ik}(t)$ on the right of the last formula line

remains bounded as $t \to \infty$, every x_k^0 being a constant. Consequently, from the last formula line and from (9) and (7),

$$\int^{\infty} | f_i(t; x_1(t), \cdots, x_n(t)) | \, dt < \infty.$$

Since $(x_1(t), \cdots, x_n(t))$ is a solution of (3), this means that the n integrals $\int^{\infty} x_i'(t)dt$ are absolutely convergent. In particular, they are convergent. But their convergence is equivalent to the existence of finite limits $x_i(\infty)$, since $x_i' = dx_i/dt$.

5. The above formulations and proofs are within the real field. However, it is clear from the proofs that *the results can immediately be transcribed to the analytic case of the complex field*, where the real t-axis is replaced by the complex z-plane. What then corresponds to the half-line $T' < t < \infty$ may, for instance, be a wedge $| \arg z | < \alpha$, and what corresponds to $t \to \infty$ is the limit process $z \to \infty$, where z is restricted to the wedge.

Needless to say, the theorems which thus result for a (partial) neighborhood of $z = \infty$ can be transformed into corresponding theorems relating to the behavior of arbitrary solutions of certain non-linear analytic differential equations near a singular point at $z = 0$.

ADDENDUM (May 30, 1945). In correcting an impression I originally had concerning the possible instability of the mapping mentioned after (i), Professor Siegel observed that, *under the assumptions of* (i), *the mapping* $(x_1(0), \cdots, x_n(0)) \to (x_1(\infty), \cdots, x_n(\infty))$ *of the x-space is continuous and one-to-one.* The proof is published here with his kind permission.

Let $u \cdot v = u_1v_1 + \cdots + u_nv_n$ and $| u | = (u \cdot u)^{\frac{1}{2}}$, where $u = x, f, \cdots$. Then (3) and (8) become $x' = f(t; x)$ and

$$| f(t; x^*) - f(t; x^{**}) | \leqq \lambda(t) | x^* - x^{**} |, \text{ where } \int^{\infty} \lambda(t)dt < \infty,$$

by (7). Let

$$r = r(t) = | x^*(t) - x^{**}(t) |,$$

where $x = x^*(t)$ and $x = x^{**}(t)$ are two solutions of $x' = f(t; x)$. Then, by Schwarz's inequality,

$$| rr' | \leqq r \, | \{x^*(t) - x^{**}(t)\}' | = r \, | f(t; x^*(t)) - f(t; x^{**}(t)) |.$$

Since the last difference is majorized by $\lambda(t)r$, it follows that $|r'|$ cannot exceed $r\lambda$ for any t. It follows, therefore, from the absolute integrability of $\lambda(t)$ that the limit, $r(\infty)$, of $r(t)$ as $t \to \infty$ has a value contained between two bounds $r(0)e^{\pm c}$, where c is independent of $r(0)$. In other words,

$$|x^*(0) - x^{**}(0)|\, e^{-c} \leqq |x^*(\infty) - x^{**}(\infty)| \leqq |x^*(0) - x^{**}(0)|\, e^{c},$$

where the existence of the limits is assured by (i).

According to (i), the mapping $x(0) \to x(\infty)$ is defined on the whole x-space. Let this mapping be denoted by μ. Then the last formula line implies that μ is continuous and has a unique inverse μ^{-1}. Hence, all that remains to be shown is that M is the whole x-space, if M denotes the μ-image of the whole x-space.

First, M is an open set, by Jordan's theorem. Next, if $x^1(\infty), x^2(\infty), \cdots$ is a convergent sequence of points of M, the last formula line implies that $x^1(0), x^2(0), \cdots$ is a bounded sequence and that, if $x(0)$ denotes a cluster point of the latter sequence, then $x(0)$ is identical with the point $\lim x^n(\infty)$ the existence of which is assumed. Accordingly, the set M is closed. Since it is open, it must be the whole space.

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[2] A. Wintner, "The non-local existence problem of ordinary differential equations," *ibid.*, vol. 67 (1945), pp. 277-284.

[3] ———, "Small perturbations," *ibid.*, pp. 417-430.

MONOTONE SERIES.*

By R. W. HAMMING.

1. Introduction. Let Σc_n be a monotone convergent series and let Σd_n be a monotone divergent series. Pringsheim has shown, contrary to first impressions, that there exist monotone series such that

$$\limsup c_n/d_n = \infty.$$

On the other hand

$$\liminf c_n/d_n = 0,$$

since otherwise $c_n/d_n \geq \epsilon$ for all n would imply $\Sigma c_n \geq \epsilon \Sigma d_n$.

The object of the present paper is to study more closely the possible behavior of c_n/d_n as n approaches infinity, in particular, the question of how many of the terms of c_n/d_n may be bounded away from zero. By replacing Σc_n by $1/\epsilon \Sigma c_n$ this is transformed into the question of how many of the c_n may be greater than the corresponding d_n. Since this may happen infinitely often, we study the function $A(n)$ which is defined as the average number of times $c_n \geq d_n$.

The basic result is given in Theorem 1 of **2**. Section **3** contains an elementary example which shows that not only may c_n be greater than or equal to d_n infinitely often, but that for certain series

$$\limsup A(n) = 1.$$

Section **4** discusses the summability of the function $A(n)$.

2. Fundamental theorem. Throughout this paper Σc_n is a monotone convergent series and Σd_n is a monotone divergent series. Let $N(n)$ be the number of terms for which $c_i \geq d_i$, $i = 1, 2, \cdots, n$. Then the function

$$A(n) = N(n)/n$$

measures the average number of times the c_i are at least as large as the corresponding d_i terms.

THEOREM 1. *For every $\epsilon > 0$ and any $k > 1$ there exist infinitely many r_i ($r_i \to \infty$) such that throughout the interval $r_i \leq n \leq k r_i$ we have $A(n) < \epsilon$.*

To prove the theorem we first develop two lemmas.

* Received April 9, 1945.

LEMMA 1. *If, for a given positive* ϵ *less than* 1, $A(n) \geq \epsilon$ *for some value or values of* n *in each of the intervals*

$$K^m \leq n \leq K^{m+\frac{1}{2}} \quad \text{and} \quad K^{m+\frac{1}{2}} \leq n \leq K^{m+1}$$

where $K \geq 4/\epsilon^2$, *then for some value* $n = n_m$ *in the combined interval*

$$K^m \leq n \leq K^{m+1} \tag{2.1}$$

both

$$A(n_m) \geq \epsilon \quad \text{and} \quad c_{n_m} \geq d_{n_m}. \tag{2.2}$$

Proof. If the first interval begins with $A(n) < \epsilon$ then the first n for which $A(n) \geq \epsilon$ will provide a suitable n_m. The same argument can be applied to the second interval, so that the lemma is proved except in the case where the first interval begins and ends with $A(n) \geq \epsilon$. In this case there must be a suitable n_m in the first interval since otherwise we would have at the end of the interval

$$A(K^{m+\frac{1}{2}}) = N(K^{m+\frac{1}{2}})/K^{m+\frac{1}{2}} = N(K^m)/K^{m+\frac{1}{2}} \leq K^m/K^{m+\frac{1}{2}} = 1/K^{\frac{1}{2}} \leq \epsilon/2.$$

LEMMA 2. *For every positive* ϵ *less than* 1 *there exist infinitely many intervals*

$$K^m \leq n \leq K^{m+\frac{1}{2}} \quad \text{and} \quad K^{m+\frac{1}{2}} \leq n \leq K^{m+1} \qquad (K \geq 4/\epsilon^2)$$

throughout which $A(n) < \epsilon$.

Proof. We first assume, contrary to what we wish to prove, that in every interval $A(n) \geq \epsilon$ for some value or values, and we apply Lemma 1 to conclude that in the combined interval (2.1) there is an n_m such that both parts of (2.2) hold. Thus, selecting every other suitable n value, that is only n_{2m}'s, and using the monotoneity of the c_n we have

$$\begin{aligned} \sum_n c_n &\geq \sum_m c_{n_{2m}}\{N(n_{2m}) - N(n_{2m-2})\} \\ &\geq \sum_m d_{n_{2m}}\{N(n_{2m}) - N(n_{2m-2})\}. \end{aligned} \tag{2.3}$$

We now introduce the function δ_m defined by

$$\delta_m = \frac{N(n_{2m}) - N(n_{2m-2})}{N(n_{2m+2}) - N(n_{2m})}.$$

By definition $A(n) = N(n)/n \leq 1$, so that $A(n_{2m}) \geq \epsilon$ becomes

$$n_{2m} \geq N(n_{2m}) \geq \epsilon n_{2m}.$$

Applying this to δ_m we have

$$\delta_m \geq \frac{\epsilon n_{2m} - n_{2m-2}}{n_{2m+2}}.$$

But n_{2m} satisfies $K^{2m} \leq n_{2m} \leq K^{2m+1}$ so that

$$\delta_m \geq \frac{\epsilon K^{2m} - K^{2m-1}}{K^{2m+3}} \geq \frac{\epsilon K - 1}{K^4} \geq \frac{4/\epsilon - 1}{K^4} \geq \frac{3}{K^4} > 0.$$

Returning to (2.3) and using the result for δ_m as well as the monotoneity of the d_n we have

$$\sum_n c_n \geq \sum_m d_{n_{2m}} \delta_m \{N(n_{2m+2}) - N(n_{2m})\} \geq (3/K^4) \sum_{n=n_2} d_n$$

which is a contradiction.

If now we assume that a finite number of the original intervals contained no suitable n values, this would lead to at most a finite number of the double intervals failing to have the properties (2.2). By deleting a finite number of terms from Σd_n to correspond to these exceptions the divergence is not affected, and hence the contradiction will still occur. Thus Lemma 2 is proved. With a change in notation it is easily seen that Theorem 1 is a special case of Lemma 2.

COROLLARY to Theorem 1. *If $A(n)$ approaches a limit, then this limit is zero.*

3. An example. The corollary discusses the case where $A(n)$ approaches a limit. That $A(n)$ need not approach a limit is shown by the following example. Indeed the example shows that while according to Theorem 1 $A(n)$ must become arbitrarily small throughout arbitrarily long intervals, nevertheless the lim sup of $A(n)$ may be 1.

For the monotone divergent series let $d_n = 1/(n \log n)$. For Σc_n set $c_n = \dfrac{2m}{2^{2^m} \log 2^{2^m}}$ whenever $2^{2^{(m-1)}} < n \leq 2^{2^m}$. Clearly Σc_n is monotone. The proof of its convergence runs as follows:

$$\Sigma c_n = \Sigma (2^{2^m} - 2^{2^{(m-1)}}) \frac{2m}{2^{2^m} \log 2^{2^m}} \leq \frac{2}{\log 2} \Sigma \frac{m}{2^m} .$$

We note that, as in Pringsheim's example, $\limsup c_n/d_n = \infty$ since for $n = 2^{2^m}$ we have $c_n/d_n = 2m$.

For n satisfying $\dfrac{1}{m} \cdot 2^{2^m} \leq n \leq 2^{2^m}$ we have

$$\frac{c_n}{d_n} \geq \frac{2m}{2^{2^m} \log 2^{2^m}} \cdot \frac{2^{2^m}}{m} \log \left(\frac{2^{2^m}}{m} \right)$$

$$\geq 2 \left\{ 1 - \frac{\log m}{2^m \log 2} \right\} \geq 1 \qquad (m \geq 1).$$

Now calculating $A(n)$ at $n = 2^{2^m}$ we have

$$A(2^{2^m}) \geq \frac{(1-(1/m))2^{2^m}}{2^{2^m}} = 1 - \frac{1}{m}$$

and the lim sup $A(n) = 1$.

4. Summability of $A(n)$. From the two given monotone series we have derived a sequence $A(n)$ which, as we have shown in **2** and **3**, may either approach zero or oscillate as n approaches infinity. One of the standard methods of treating an oscillating sequence is to calculate its Cesàro-Hölder means. Since the Cesàro and Hölder methods are equivalent we may use the Hölder method in proving the following general result.

THEOREM 2. *For every $\epsilon > 0$ and any $k > 1$ there exist infinitely many r_i ($r_i \to \infty$) such that throughout the interval $r_i \leq n \leq kr_i$ we have $C^{(m)}(n) < \epsilon$, where $C^{(m)}(n)$ is the m-th Cesàro-Hölder mean.*

Proof. The proof is by induction with Theorem 1 as a basis, m being zero. Thus we assume that we can pick any E and K and find infinitely many R_i ($R_i \to \infty$) such that

$$C^{(m-1)}(n) < E \quad \text{for} \quad R_i \leq n \leq KR_i.$$

Pick $E = \epsilon/2$, $K = 2k/\epsilon$, and set $\alpha = 1/k$. Thus, since $C^{(m-1)}(n) \leq 1$ for all n,

$$C^{(m)}(\alpha KR_i) \leq \frac{R_i + E(\alpha K - 1)R_i}{\alpha KR_i} = \frac{1}{\alpha K} + E\left(1 - \frac{1}{\alpha K}\right) < \epsilon/2 + \epsilon/2 = \epsilon$$

and we have for all n satisfying $\alpha KR_i \leq n \leq KR_i$

$$C^{(m)}(n) < \epsilon.$$

Setting $R_i = \epsilon r_i/2$ the above interval becomes

$$\alpha KR_i = \frac{1}{k} \cdot \frac{2k}{\epsilon} \cdot \frac{\epsilon r_i}{2} \leq n \leq \frac{2k}{\epsilon} \cdot \frac{\epsilon r_i}{2} = KR_i$$

or

$$r_i \leq n \leq kr_i \qquad (r_i \to \infty).$$

COROLLARY to Theorem 2. *If $A(n)$ is summable $C^{(m)}(n)$ then the sum is zero.*

In closing it is to be noted that examples, only slightly more complicated than that in **3**, can be given to show that $A(n)$ may be summable $C^{(m)}(n)$ and not $C^{(m-1)}(n)$, and thus Theorem 2 is not vacuous.

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CONFORMAL MAPS WITH ISOTHERMAL SYSTEMS OF SCALE CURVES.*

By JOHN DE CICCO.

1. Introduction. Let a surface Σ be mapped in a point-to-point fashion upon a plane π. The *scale function* $\sigma = ds/dS$, which is the ratio of the differentials of arc lengths on the plane π and the surface Σ, depends upon the position of the point and also upon the direction through the point. It is independent of the direction through the point if, and only if, the mapping of Σ upon π is conformal. The scale σ is a constant only in the degenerate situation where Σ is developable and the mapping is an unrolling of Σ upon π, followed by a similitude in π.

A *scale curve* is the locus of a point along which the scale σ does not vary. For a non-conformal map of Σ upon π, there are ∞^2 scale curves. Under a conformal map, there are ∞^1 scale curves. In the degenerate situation mentioned above, every curve is a scale curve. Henceforth we shall exclude this case from consideration so that the scale function σ is a non-constant function.

In the present paper, we shall be concerned chiefly with conformal maps. Hence σ is a non-constant function which depends only upon the position of the point; and there are ∞^1 scale curves. It is remarked that any family of ∞^1 scale curves in the plane π may represent the scale curves of a conformal map of some surface Σ upon the plane π. In particular, any isothermal system can be the scale curves of a conformal map of some surface Σ upon π. We shall consider the surfaces Σ applicable upon a surface of revolution for which there exists a conformal map of Σ upon the plane π with an isothermal family of scale curves, which are neither parallel straight lines nor concentric circles.

Since we shall be mainly concerned with surfaces applicable upon surfaces of revolution, it is convenient to introduce the following terminology. A *sphere-like surface* is any surface of constant, but not zero, gaussian curvature. A *vase-like surface* is any surface which is applicable upon a surface of revolution of variable gaussian curvature. Thus a surface which is applicable upon a surface of revolution may be either developable, or sphere-like, or vase-like. These three classes are mutually exclusive.

Obviously there always exists a developable surface Σ for which any isothermal system of curves may represent the scale curves of a conformal map

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of Σ upon the plane π. We shall omit this case from consideration so that henceforth the gaussian curvature is assumed to be not identically zero.

Of course, any sphere-like or vase-like surface Σ may be conformally mapped upon the plane π such that the scale curves form an isothermal family. For any such Σ the scale curves are parallel straight lines or concentric circles. For a sphere-like surface we have proved that these are the only possible conformal representations on π with an isothermal family of scale curves. *The main problem of this paper is to discover the forms of the linear-elements of all vase-like surfaces Σ for which there exist conformal maps upon a plane π with isothermal systems of scale curves which are neither parallel straight lines nor concentric circles.* It is found that there are essentially *seven* types in the real domain, or *six* types in the complex domain.

For our work, we shall obtain, in the case of a conformal representation, the analytic form of a theorem of Kasner which states that a surface is applicable upon a surface of revolution if, and only if, it possess an isothermal system of geodesics.[1]

2. Preliminary formulas. Consider any conformal map of a surface Σ upon a plane π with minimal coordinates $(u = x + iy,\ v = x - iy)$. The linear-element of Σ is

$$ds^2 = 2e^{\lambda(u,v)}dudv. \tag{1}$$

The gaussian curvature G is defined by the formula

$$G = -e^{-\lambda}\lambda_{uv}, \tag{2}$$

and its partial derivatives of the first order with respect to u and v are

$$G_u = -e^{-\lambda}(\lambda_{uuv} - \lambda_u\lambda_{uv}), \qquad G_v = -e^{-\lambda}(\lambda_{uvv} - \lambda_v\lambda_{uv}). \tag{3}$$

By (2), it is seen that the surface Σ is developable if, and only if, λ is harmonic, that is, $\lambda_{uv} = 0$. From (3), it follows that Σ is sphere-like if, and only if, λ satisfies the system of two partial differential equations of the third order: $\lambda_{uuv} - \lambda_u\lambda_{uv} = 0$, $\lambda_{uvv} - \lambda_v\lambda_{uv} = 0$.

The differential equation of the ∞^2 geodesics of the surface Σ is

$$v'' = v'(\lambda_u - v'\lambda_v). \tag{4}$$

3. The analytic conditions in a conformal representation for a vase-

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[1] Kasner, "Isothermal systems of geodesics," *Transactions of the American Mathematical Society*, vol. 5 (1904), pp. 55-60. See also De Cicco, "New proofs of the theorems of Beltrami and Kasner on linear families," *Bulletin of the American Mathematical Society*, vol. 49 (1943), pp. 407-412.

like surface. The linear-element (1) represents a vase-like surface Σ if, and only if, λ is given by an equation of the form

$$\lambda = F[\phi(u) + \psi(v)] + \log \phi_u \psi_v, \tag{5}$$

where $\phi_u \neq 0$ and $\psi_v \neq 0$. This means that $[\phi + \psi]$ and $[\lambda - \log \phi_u \psi_v]$ are functionally dependent. Hence

$$\lambda_u/\phi_u - \lambda_v/\psi_v = \phi_{uu}/\phi_u^2 - \psi_{vv}/\psi_v^2. \tag{6}$$

Therefore it remains to determine the conditions under which a function λ can satisfy an equation of this form.

Obtaining the cross derivative with respect to u and v and using this equation again, we find by (3) that, since the gaussian curvature G is not constant, there exists a function $\rho(u, v)$ such that

$$\phi_u = \rho G_u, \qquad \psi_v = \rho G_v. \tag{7}$$

Since ϕ depends upon u only and ψ depends upon v only, the function ρ must satisfy the conditions

$$(\partial/\partial u) \log \rho = -(\partial/\partial u) \log G_v, \qquad (\partial/\partial v) \log \rho = -(\partial/\partial v) \log G_u. \tag{8}$$

The compatibility condition for ρ yields the equation

$$(\partial^2/\partial u \partial v) \log G_u/G_v = 0. \tag{9}$$

This means that $\log G_u/G_v$ is a harmonic function, or that the gaussian curvature G is a function of a harmonic function.

Next substituting (7) into (6), we discover that

$$\lambda_u/G_u - \lambda_v/G_v = G_{uu}/G_u^2 - G_{vv}/G_v^2. \tag{10}$$

By equations (9) and (10), we have obtained the following analytic form of Kasner's theorem characterizing surfaces Σ applicable upon surfaces of revolution of variable gaussian curvature.

THEOREM 1. *A surface Σ with linear-element (1) is vase-like if, and only if, the system of two partial differential equations of the fourth order*

$$\begin{aligned} G_{uuv}/G_u - G_{uvv}/G_v &= G_{uv}(G_{uu}/G_u^2 - G_{vv}/G_v^2), \\ \lambda_u/G_u - \lambda_v/G_v &= G_{uu}/G_u^2 - G_{vv}/G_v^2, \end{aligned} \tag{11}$$

is identically satisfied.

Geometrically this result means that the ∞^1 curves $v' = G_u/G_v$ are an isothermal system of geodesics, as may be verified by substituting into the differential equation (4) of geodesics, and into the condition for isothermal

families. This isothermal system of geodesics of Σ will correspond to the meridians of the isometric surface of revolution. The parallels correspond to the curves $G = \text{const.}$, and also form an isothermal system.

THEOREM 2. *In any vase-like surface Σ which is conformally mapped upon a plane π, if the scale curves $\lambda = const.$, coincide with the family $G = const.$, then the scale curves are parallel straight lines or concentric circles in π. If $G = const.$, form a parallel family in the plane π, they coincide with the scale curves $\lambda = const.$, and hence are parallel straight lines or concentric circles.*

This is a consequence of the conditions (11) for a vase-like surface Σ and some theorems developed in one of our preceding papers.[2]

4. Vase-like surfaces Σ for which conformal maps exist such that the scale curves form isothermal systems which are neither parallel straight lines nor concentric circles. The reasons why we restrict ourselves to vase-like surfaces are the following. Let Σ be any surface applicable upon a surface of revolution. Then Σ is either developable, or sphere-like, or vase-like.

If Σ is developable, then in any conformal map of Σ upon a plane π, the scale curves form an isothermal family. Conversely any isothermal system of curves can serve as the scale curves of a conformal map of some developable surface Σ upon a plane π.

We have proved elsewhere that if a sphere-like surface Σ is conformally represented upon a plane π such that the scale curves form an isothermal family, then the scale curves must be either parallel straight lines or concentric circles.

The preceding statements demonstrate the reasons why we must restrict ourselves to vase-like surfaces. Also the isothermal systems of scale curves must be neither parallel straight lines nor concentric circles, for it is well-known that any surface Σ applicable upon a surface of revolution may be conformally mapped upon a plane π such that the scale curves are either parallel straight lines or concentric circles.

In the remainder of the paper, we shall prove the following result.

FUNDAMENTAL THEOREM 3. *The linear-elements of all the vase-like sur-*

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[2] Kasner and De Cicco, "Scale curves in cartography," *Science*, vol. 98 (1943), pp. 324-325. Also "Scale curves in conformal maps," *Proceedings of the National Academy of Sciences*, vol. 30 (1944), pp. 162-164, and "Scale curves in general cartography," *Proceedings of the National Academy of Sciences*, vol. 30 (1944), pp. 211-215. Finally see "Geometry of scale curves in conformal maps," *American Journal of Mathematics*, vol. 67 (1945), pp. 157-166.

faces for which conformal maps exist such that the scale curves form isothermal systems which are neither parallel straight lines nor concentric circles can be reduced to the following types by appropriate conformal transformations

$$(12.1)\quad ds^2 = \frac{ce^{uv}}{(uv)^n}\,du\,dv,$$

$$(12.2)\quad ds^2 = \frac{2e^{f(u)+g(v)}}{(u+v)^2}\,du\,dv,$$

$$(12.3)\quad ds^2 = \frac{[(au+b)(cv+d)]^{n-2}}{(u+v)^n}\,du\,dv,\ \text{where } n \neq 0, 2,$$

$$(12.4)\quad ds^2 = \frac{ce^{u+v}}{(u+v)^n}\,du\,dv,\ \text{where } n \neq 0, 2,$$

$$(12.5)\quad ds^2 = \frac{c(uv)^{(n-2)/2}}{(u+v)^n}\,du\,dv,\ \text{where } n \neq 0, 2,$$

$$(12.6)\quad ds^2 = \frac{c(uv)^k}{(uv-1)^n}\,du\,dv,\ \text{where } k \neq 0,\ n-2,\ \text{and } n \neq 0, 2,$$

$$(12.7)\quad ds^2 = \frac{c[u^{1+ib}v^{1-ib}]^{(n-2)/2}}{(u+v)^n}\,du\,dv,\ \text{where } b \neq 0,\ \text{and } n \neq 0, 2,$$

where $c \neq 0$, n, a, b, d, k, are constants.

In the real domain, there are essentially seven types of such vase-like surfaces, but in the imaginary domain there are only six types; the last two types, namely (12.6) and (12.7), can be made equivalent under an imaginary conformal transformation.

5. The beginning of the proof of our Fundamental Theorem 3. Any surface Σ, not necessarily vase-like, for which a conformal map exists such that the scale curves form an isothermal system has the linear-element of the form

$$(13)\qquad ds^2 = 2e^{\lambda[\alpha(U)+\beta(V)]}dU\,dV,$$

where $\alpha_U \neq 0$ and $\beta_V \neq 0$. Let $u = \alpha(U)$, $v = \beta(V)$. Noting that this transformation may be defined by $\log dU/du = \phi(u)$, $\log dV/dv = \psi(v)$, it is seen that the linear-element (13) may be written in the form

$$(14)\qquad ds^2 = 2e^{\lambda(w)+\phi(u)+\psi(v)}du\,dv,$$

where $w = u + v$. This is the form of the linear-element of that class of surfaces Σ for which there exists a conformal map of Σ upon a plane π such that the scale curves form an isothermal system.

The gaussian curvature of the surface Σ with the linear-element (14) is $G = -e^{-\lambda-\phi-\psi}\lambda_{ww}$. The partial derivatives of the first order with respect to u and v of G are

$$(15) \qquad G_u = -e^{-\lambda-\phi-\psi}\lambda_{ww}(g-\phi_u), \qquad G_v = -e^{-\lambda-\phi-\psi}\lambda_{ww}(g-\psi_v),$$

where

$$(16) \qquad g = g(w) = \lambda_{www}/\lambda_{ww} - \lambda_w.$$

Since $G_u \neq 0$ and $G_v \neq 0$, we see that $\lambda_{ww} \neq 0$, $g \neq \phi_u$ and $g \neq \psi_v$. We omit the case where $\phi_u = \psi_v$, because otherwise we have the cases where the scale curves are parallel straight lines or concentric circles.

The condition that the curves $G = \text{const.}$, be an isothermal family reduces to

$$(17) \qquad (\partial^2/\partial u \partial v) \log (G_u/G_v) = (\partial^2/\partial u \partial v) \log (g-\phi_u)/(g-\psi_v) = 0.$$

The other condition that the orthogonal trajectories of $G = \text{const.}$, be a system of geodesics becomes

$$(18) \qquad (\lambda_w + g)/(g-\phi_u) - (\lambda_w + g)/(g-\psi_v) = (g_w - \phi_{uu})/(g-\phi_u)^2 - (g_w - \psi_{vv})/(g-\psi_v)^2.$$

Upon expanding (17) and making use of (18), we discover, since $\phi_u \neq \psi_v$, that

$$(19) \qquad g_{ww} = g_w \lambda_{www}/\lambda_{ww}.$$

From this, it is seen that there exist constants (a, b) such that

$$(20) \qquad g = (2a-1)\lambda_w + 2b.$$

Substituting this into (16), we find

$$(21) \qquad \lambda_{ww} = a\lambda_w^2 + 2b\lambda_w + c,$$

where (a, b, c) are constants, not all zero. This condition is only necessary for the solution of our problem.

If $a = 0$, we shall prove that the linear-element can be reduced to the form

$$(22) \qquad (\text{I}): \quad ds^2 = 2e^{uv+f(u)+g(v)}du\,dv.$$

If $a = b = 0$, then $c \neq 0$ since $\lambda_{ww} \neq 0$. Thus λ is a quadratic function of $w = u + v$. By an appropriate similitude in π, the linear-element (14) can be reduced to the form (I).

If $a = 0$ and $b \neq 0$, it is found by (21) that λ is given by an equation of the form

$$(23) \qquad \lambda = -\frac{cw}{2b} + h^2 e^{2bw} + k,$$

where $h \neq 0$ and k are constants. The transformation $U = he^{2bu}$, $V = he^{2bv}$, reduces this to the form (I).

If $a \neq 0$, we shall prove that the linear-element can be reduced to the form

$$(24) \qquad (\text{II}): \quad ds^2 = 2\,\frac{e^{f(u)+g(v)}}{(u+v)^n}\,du\,dv,$$

where n is a non-zero constant.

If $a \neq 0$ and $b^2 - ac = 0$, we find that λ is given by

$$(25) \qquad \lambda = -(b/a)(w - w_1) - (1/a)\log(w - w_0),$$

where (w_0, w_1) are constants. Let $n = 1/a$ and let $U = u - u_0$, $V = v - v_0$. Hence the linear-element (14) in this case is reduced to the form (II).

Finally let $a \neq 0$ and $b^2 - ac \neq 0$. The complete solution of (21) is

$$(26) \qquad \lambda = -(b/a)(w - w_1) - (1/a)\log\left[e^{d(w-w_0)} - e^{-d(w-w_0)}\right],$$

where $d^2 = b^2 - ac$, w_0, w_1, are constants. Hence the linear-element (14) in this case may be written as

$$(27) \qquad ds^2 = \frac{2e^{f(u)+g(v)}}{\left[e^{2d(w-w_0)} - 1\right]^n}\,du\,dv,$$

where $n = 1/a \neq 0$. Now let

$$(28) \quad (2U+1)/(2U-1) = e^{2d(v-v_0)}, \quad (2V+1)/(2V-1) = e^{2d(u-u_0)}.$$

(If d is pure imaginary, replace the ± 1 in the first equation by $\pm i$ and the ± 1 in the second equation by $\pm i$). Substituting (28) into (27), we see that the linear-element (14) is changed to the form (II).

6. The discussion of the Case (I) where the linear-element of the surface Σ is given by (22). The gaussian curvature is $G = -e^{-uv-f-g}$. The condition that $G = \text{const.}$, be an isothermal family becomes

$$(29) \qquad f_{uu}/(v + f_u)^2 = g_{vv}/(u + g_v)^2.$$

The other condition reduces to this also.

If f_{uu} or g_{vv} is zero, then both are zero. Hence f and g are both linear functions in u and v separately. Under an appropriate translation, we find that case (I) in this instance can be reduced to (12.1), where $n = 0$, of our Fundamental Theorem 3.

If f_{uu} or g_{vv} is not zero, then both are not zero. The cross derivative of the square root of the reciprocal of (29) is

$$(30) \qquad (\partial/\partial u)\,(f_{uu})^{-\frac{1}{2}} = \pm\,(\partial/\partial v)\,(g_{vv})^{-\frac{1}{2}}.$$

This demonstrates that

$$(31) \qquad f_u = -a^2/(u-u_0) + b, \qquad g_v = -a^2/(v-v_0) + c,$$

where $(a \neq 0, b, c)$ are constants. The substitution of (31) into (29) shows that $c = -u_0$, $b = -v_0$. Hence f and g are of the forms

$$(32) \quad f = -a^2 \log (u-u_0) - v_0 u + u_1, \quad g = -a^2 \log (v-v_0) - u_0 v + v_1,$$

where $(a \neq 0, u_0, v_0, u_1, v_1)$ are constants. Under an appropriate translation, our linear-element can be reduced to the form (12.1) of our Fundamental Theorem 3. In the real domain, n is a positive real constant.

7. The discussion of the Case (II) where the linear-element of the surface Σ is given by (24). The gaussian curvature is $G = -n(u+v)^{n-2} \times e^{-f(u)-g(v)}$. Note that $n \neq 0$. The condition that $G = \text{const.}$, be an isothermal family is

$$(33) \qquad \frac{(n-2)f_{uu} + f_u^2}{[(u+v)f_u - (n-2)]^2} = \frac{(n-2)g_{vv} + g_v^2}{[(u+v)g_v - (n-2)]^2}.$$

The other condition that the orthogonal trajectories of $G = \text{const.}$, form a geodesic system may be shown to be equivalent to this.

If $n = 2$, the above equation is an identity yielding the linear-element (12.2) of our Fundamental Theorem 3.

Note that $n = 0$ yields a developable surface, which case is excluded from consideration. Henceforth $n \neq 0, 2$.

Let $f_u = g_v = 0$. The linear-element in this case assumes the form (12.3), where $a = c = 0$.

In the imaginary domain, we may have the case where $f_u = 0$ but $g_v \neq 0$; or $f_u \neq 0$ but $g_v = 0$. Here we have the linear-element (12.3), where $a = 0$ or $c = 0$.

If $n \neq 0, 2$, and if either of the numerators of (33) is zero but $f_u \neq 0$ and $g_v \neq 0$, it follows that in this case the linear-element is of the form (12.3) where $a \neq 0$ and $c \neq 0$.

Finally we consider the case where either numerator of (33) is not zero. Then both numerators are not zero. Then the cross derivative of the square root of the reciprocal of (33), is

$$(34) \qquad \frac{\partial}{\partial u} \frac{f_u}{[(n-2)f_{uu} + f_u^2]^{\frac{1}{2}}} = \pm \frac{\partial}{\partial v} \frac{g_v}{[(n-2)g_v + g_v^2]^{\frac{1}{2}}}.$$

Hence each side represents the same constant.

First we must consider the case when this constant is zero. We find

$$(35) \qquad (n-2)f_{uu}/f_u^2 = a^2 - 1, \qquad (n-2)g_{vv}/g_v^2 = b^2 - 1,$$

where a and b are constants. Substituting (35) into (33), we find

$$(36) \qquad \frac{a^2 f_u^2}{[(u+v)f_u - (n-2)]^2} = \frac{b^2 g_v^2}{[(u+v)g_v - (n-2)]^2}.$$

Since we are working under the condition where neither numerator of (33) is zero, we see that $a \neq 0$, $b \neq 0$, $f_u \neq 0$, $g_v \neq 0$. Taking the cross derivative of the reciprocal of (36) and making use of (36) we find that $a^2 = b^2$. From (36), we obtain

(37) Either $f_u = g_v$, or $(n-2)/f_u + (n-2)/g_v - 2(u+v) = 0$.

If $n \neq 2$, $f_u \neq 0$, $g_v \neq 0$ and $f_u = g_v$, it is found that under an appropriate similitude the linear-element of Σ can be reduced to the form (12.4) of our Fundamental Theorem 3.

Next if the second of the conditions (37) is valid, it is found, by (35), that $b^2 = a^2 = -1$. Thus we find by (35) and the second of the conditions (37) that

$$(38) \qquad \begin{aligned} f &= [(n-2)/2] \log (u - u_0) + \text{const.}, \\ g &= [(n-2)/2] \log (v + u_0) + \text{const.} \end{aligned}$$

By an appropriate translation, we find that if $n \neq 0, 2$, and if the equation (35) together with the second of the conditions (37), hold, the linear-element can be reduced to the form (12.5).

Finally we have to consider the case where the equal constant of (34) is not zero. Hence we find

$$(39) \qquad \frac{(n-2)f_{uu} + f_u^2}{f_u^2} = \frac{a^2}{(u-u_0)^2}, \quad \frac{(n-2)g_{vv} + g_v^2}{g_v^2} = \frac{a^2}{(v-v_0)^2},$$

where $(a \neq 0, u_0, v_0)$ are constants. Substitute this into (33) and we find

$$(40) \qquad \frac{f_u^2}{(u-u_0)^2[(u+v)f_u - (n-2)]^2} = \frac{g_v^2}{(v-v_0)^2[(u+v)g_v - (n-2)]^2}.$$

From this equation, it follows that

$$(41) \qquad (u-u_0)[(u+v) - (n-2)/f_u] = \pm (v-v_0)[(u+v) - (n-2)/g_v].$$

Differentiating this equation with respect to v and also with respect to u and making use of (39), we discover that the minus sign is impossible and we get

$$(42) \qquad f_u = \frac{(n-2)(u-u_0)}{(u+v_0)(u-u_0) + a^2}, \quad g_v = \frac{(n-2)(v-v_0)}{(v+u_0)(v-v_0) + a^2}.$$

The equation (33) is obviously satisfied by these.

For this general case, let us apply the translation $U = u + \frac{1}{2}(-u_0 + v_0)$, $V = v + \frac{1}{2}(u_0 - v_0)$. Of course, the linear-element of Σ is of the same form (II) but f and g are now given by

$$(43) \qquad f_u = \frac{(n-2)(u-\alpha)}{u^2 - b^2}, \qquad g_v = \frac{(n-2)(v-\alpha)}{v^2 - b^2},$$

where $\alpha = \frac{1}{2}(u_0 + v_0)$ and $b^2 = \alpha^2 - a^2$. Since $a \neq 0$, it follows that b cannot be equal to $\pm \alpha$.

If $b = 0$, then $\alpha \neq 0$, and we find that f and g are given by

$$(44) \quad f = (n-2)(\log u + \alpha/u) + \text{const.}, \quad g = (n-2)(\log v + \alpha/v) + \text{const.}$$

Substituting these into Case (II) and then using the substitution $U = 1/u$, $V = 1/v$, we see that our linear-element is reduced to the form (12.4) of our Fundamental Theorem 3.

Finally if $b \neq 0$, then f and g are given by

$$(45) \quad \begin{aligned} f &= [(n-2)/2b][(b-\alpha)\log(u-b) + (b+\alpha)\log(u+b)] + \text{const.} \\ g &= [(n-2)/2b][(b-\alpha)\log(v-b) + (b+\alpha)\log(v+b)] + \text{const.} \end{aligned}$$

It follows that the linear-element of Σ is of the form

$$(46) \quad ds^2 = [c/(u+v)^n][\{(u-b)(v-b)\}^{(1-a/b)} \times \{(u+b)(v+b)\}^{(1+a/b)}]^{(n-2)/2} du\, dv,$$

where $a \neq 0$, $b \neq 0$, and $\alpha \neq \pm b$.

If b is real, the transformation:

$$(47) \qquad u = b(U-1)/(U+1), \qquad v = b(V-1)/(V+1),$$

will change the form of (46) into the form (12.6).

On the other hand if b is pure imaginary, the transformataion:

$$(48) \qquad = b(U-1)/(U+1), \qquad v = -b(V-1)/(V+1),$$

will change the form (46) into the form (12.7).

This completes the proof of our Fundamental Theorem 3.

8. Conclusion. It is remarked that the conformal maps with isothermal systems of scale curves of the vase-like surfaces of our Fundamental Theorem 3, can be obtained by compounding inversely the various correspondences which were used in order to reduce the linear-element (13) to these linear-elements (12).

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COMPLEX FUNCTIONS POSSESSING DIFFERENTIALS.*

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I. THE DIFFERENTIALS OF FUNCTIONS.

1. Introduction. We shall be interested in the complex function $f(z, z')$ of two complex variables z and z'. We shall also consider incidentally the polygenic function $f(z)$ obtained by replacing z' by $\bar{z}$ the conjugate of z. It is polygenic in the sense that its z derivative at a point depends on the direction of approach to the point.

2. Purpose and content. The purpose of this paper is to study differentials of complex functions, in particular the restricted Hamilton [1] and the Young [2] differentials and their relations to each other. Closely related to the restricted Hamilton differential are the Stolz [3] and the Rainich [4] differentials; the latter is a Hamilton differential restricted by the linearity property and in addition a property something less than continuity.

The Young definition is also given by Mrs. Young [5] in discussing functions possessing differentials. In her paper necessary and sufficient conditions for the existence of a Young differential of a complex function are given. However, some of these conditions, to say the least. are superfluous, and properly formulated the theorem is not proved.

The necessary and sufficient conditions for the existence of a Young differential will be set up here and proved. Also certain results of the Fundamenta paper will be generalized to the complex plane. Finally, a power-series expansion of a polygenic function will be given.

3. Regularity. The Hamilton differential was invented to make the absolute geometric development of vector analysis possible; the differential quotient or derivative of a point function is non-existent since division by a

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[1] V. C. Poor, "On the Hamilton differential," *Bulletin of the American Mathematical Society*, vol. 51 (1945), pp. 945-948.

[2] W. H. Young, "On differentials," *Proceedings of the London Mathematical Society*, Series 2, vol. 7 (1908), p. 162.

[3] O. Stolz, *Differential und integral Rechnung*, vol. I, p. 132.

[4] G. Y. Rainich, *American Journal of Mathematics*, vol. 46 (1924), p. 78.

[5] Grace Chisholm Young, "On functions possessing differentials," *Fundamenta Mathematicae*, Tome 15 (1930), pp. 61-94.

vector is excluded. This definition will be restricted by a linearity condition making it essentially equivalent to the Stolz differential. For brevity we shall call a function regular if it possesses a differential, R-regular if it possesses a restricted Hamilton differential and Y-regular if it possesses a Young differential.

4. The restricted Hamilton differential. The Hamilton differential may be expressed by the equation

$$f'(P_1, dP) = \lim_{\lambda \to 0} \frac{f(P_1 + \lambda dP) - f(P_1)}{\lambda} \tag{4.1 a}$$

where P_1 is a point of the two-dimensional vector space z, z', while dP is an arbitrary displacement of P_1. The restricted Hamilton differential is a Hamilton differential (4.1 a) which satisfies the requirement that it be linear. The linearity property is given by

$$f'(P_1, dz\mathbf{i} + dz'\mathbf{j}) = f'(P_1, \mathbf{i})dz + f'(P_1, \mathbf{j})dz' \tag{4.1 b}$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the complex vector space.

THEOREM 1. *The necessary and sufficient condition for the complex function $f(z, z')$ to be R-regular at a point $P_1 \equiv (z_1, z'_1)$ is that $f(z, z')$ be expressible in the form*

$$f(z, z') = f(z_1, z'_1) + \frac{\partial f}{\partial z_1}(z - z_1) + \frac{\partial f}{\partial z'_1}(z' - z'_1) + \eta_1 \tag{4.2}$$

where η_1 is an infinitesimal function of $z - z_1$ and $z' - z'_1$ of an order higher than the first.

In proving the necessity of the condition one observes that

$$f(P_1 + \lambda dP) = f(P_1) + \lambda f'(P_1, dP) + \eta_1$$

follows from (4.1 a), while the linearity condition (4.1 b) changes this equation into

$$f(P_1 + \lambda dP) = f(P_1) + f'(P_1, \mathbf{i})\lambda dz + f'(P_1, \mathbf{j})\lambda dz' + \eta_1 \tag{4.3}$$

where η_1 is an infinitesimal function of λ of order higher than the first. Evidently (4.3) may be put into the coordinate form

$$f(z_1 + \lambda dz, z'_1 + \lambda dz') = f(z_1, z'_1) + f'(P_1, \mathbf{i})\lambda dz + f'(P_1, \mathbf{j})\lambda dz' + \eta_1$$

and when $z - z_1$ and $z' - z'_1$ replace λdz and $\lambda dz'$ respectively, this last equation becomes

$$f(z, z') = f(z_1, z'_1) + f'(P_1, \mathbf{i})(z - z_1) + f'(P_1, \mathbf{j})(z' - z'_1) + \eta_1$$

where η_1 is now an infinitesimal function of $z - z_1$ and $z' - z'_1$ of higher order than the first. That $f'(P_1, \mathfrak{i})$ and $f'(P_1, \mathfrak{j})$ are the partial derivatives is shown by expanding the right member of (4. 1 a), when expressed in coordinate form. Thus,

$$\begin{aligned}
&\operatorname*{limit}_{\lambda \to 0} \frac{f(z_1 + \lambda dz, z'_1 + \lambda dz') - f(z_1, z'_1)}{\lambda} \\
&\quad = \operatorname*{limit}_{\lambda \to 0} \frac{f(z_1 + \lambda dz, z'_1 + \lambda dz') - f(z_1, z'_1 + \lambda dz')}{\lambda dz} dz \\
&\qquad + \frac{f(z_1, z'_1 + \lambda dz') - f(z_1, z'_1)}{\lambda dz'} dz \\
&\quad = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial z'} dz'.
\end{aligned}$$

and when this is identified with the right member of (4. 1 b) the statement is proved and also the necessity of the condition.

The proof of the sufficiency is very simple. Our hypothesis is given by (4. 2). Using the results obtained (4. 3) follows when $z - z_1$ and $z' - z'_1$ are replaced by λdz and $\lambda dz'$ respectively in (4. 2), making η_1 an infinitesimal function of λ of higher order than the first. Thus when $f(P_1)$ is transposed and a division by λ made in (4. 3), one finds in the limit that

$$f'(P_1, dP) = f'(P_1, \mathfrak{i})dz + f'(P_1, \mathfrak{j})dz'$$

which shows that the Hamilton definition (4. 1 a) is satisfied and also that the linearity restriction (4. 1 b) holds.

Another set of necessary and sufficient conditions for R-regularity is the existence and finiteness of the partial derivatives $\partial f/\partial z_1$, $\partial f/\partial z'_1$. This is a consequence of Theorem 2. However, Theorem 1, which implies (4. 2), also implies the existence and finiteness of these partial derivatives. The conditions are thus necessary.

For the sufficiency proof one can construct (4. 2) from the hypothesis, since $\partial f/\partial z_1$ and $\partial f/\partial z'_1$ imply the continuity of $f(z, z'_1)$ and $f(z_1, z')$ at (z_1, z'_1) and thus the existence and finiteness of f at (z_1, z'_1). But (4. 2) implies R-regularity. It may be observed that $f(z, z'_1)$ and $f(z_1, z')$ are each analytic at (z_1, z'_1). Incidentally the theorem makes the restricted Hamilton and the Stolz differentials essentially equivalent.

5. The Young differential. This differential is given in the Fundamenta paper as follows:

If

$$\begin{aligned}
(5.1) \qquad &f(z_1 + \lambda dz, z'_1 + \mu dz') - f(z_1, z'_1) \\
&\qquad = [A + e_1(\lambda, \mu)]\lambda dz + [B + e_2(\lambda, \mu)]\mu dz'
\end{aligned}$$

where A and B are constants depending on the values of z_1 and z'_1 and λ and μ are arbitrary real scalar parameters, while $e_1(\lambda, \mu)$ and $e_2(\lambda, \mu)$ functions of z and z' are infinitesimal functions of λ and μ, going to zero when λ and μ are made to go to zero in any way whatever; then $f(z, z')$ is said to possess a differential at $P_1 = (z_1, z'_1)$, which may be expressed in the form $Adz + Bdz'$.

In this form the factors λ and μ are thought of as absorbed by dz and dz' respectively since dz and dz' are just as arbitrary without these factors. Also, if, in the complex function $f(z, z')$, z and z' are regarded as real variables the definition just given is exactly the one given in the Fundamenta paper aside from a slight change of notation i. e. λdz and $\mu dz'$ replace h and k respectively. Thus the complex functions, treated there, are included in $f(z, z')$.

THEOREM 2. *For the complex function $f(z, z')$ to be Y-regular at $P_1 = (z_1, z'_1)$ it is necessary and sufficient that*

(5. 2 a) $\partial f/\partial z_1$ *and* $\partial f/\partial z'_1$ *exist and are finite, and*

(5. 2 b) $$\lim_{\lambda,\mu\to 0} \phi(\lambda, \mu)/\lambda = 0; \qquad \lim_{\lambda,\mu\to 0} \phi(\lambda, \mu)/\mu = 0$$

where by definition

(5. 3) $$\phi(\lambda, \mu) \equiv f(z_1 + \lambda dz, z'_1 + \mu dz') - f(z_1, z'_1 + \mu dz') - f(z_1 + \lambda dz, z'_1) + f(z_1, z'_1)$$

To prove the necessity of the primary conditions (5. 2 a) we let μ go to zero first in (5. 1); this makes

$$f(z_1 + \lambda dz, z'_1) - (z_1, z'_1) = [A + e_1(\lambda, 0)]\lambda dz.$$

But

$$\lim_{\lambda\to 0} \frac{f(z_1 + \lambda dz, z'_1) - f(z_1, z'_1)}{\lambda dz} = \lim_{\lambda\to 0} A + e_1(\lambda, 0)$$

or

$$\partial f/\partial z_1 = A$$

since $e_1(\lambda, 0)$ goes to zero with λ by hypothesis. In a similar way we find by letting λ go to zero first in (5. 1) that

$$\partial f/\partial z'_1 = B.$$

To prove the necessity of the secondary conditions (5. 2 b) we use the results just obtained to exhibit explicitly e_1 and e_2 for any complex function. In fact (5. 1) may be written in the form

$$(5.4)\quad f(z_1+\lambda dz, z'_1+\mu dz') - f(z_1, z'_1)$$
$$\equiv \phi(\lambda,\mu) + f(z_1+\lambda dz, z'_1) - f(z_1, z'_1) + f(z_1, z'_1+\mu dz') - f(z_1, z'_1)$$
$$\equiv \left[\tau\frac{\phi(\lambda,\mu)}{\lambda dz} + A - \frac{\partial f}{\partial z_1} + \frac{f(z_1+\lambda dz, z'_1) - f(z_1, z'_1)}{\lambda dz}\right]\lambda dz$$
$$+ \left[\sigma\frac{\phi(\lambda,\mu)}{\mu dz'} + B - \frac{\partial f}{\partial z'_1} + \frac{f(z_1, z'_1+\mu dz') - (z_1, z'_1)}{\mu dz'}\right]\mu dz',$$

where $\tau+\sigma=1$. From this last form we read off e_1 and e_2:

$$(5.5)\quad e_1(\lambda,\mu) \equiv \tau\frac{\phi(\lambda,\mu)}{\lambda dz} - \frac{\partial f}{\partial z_1} + \frac{f(z_1+\lambda dz, z'_1) - f(z_1, z'_1)}{\lambda dz}$$
$$e_2(\lambda,\mu) \equiv \sigma\frac{\phi(\lambda,\mu)}{\mu dz'} - \frac{\partial f}{\partial z'_1} + \frac{f(z_1, z'_1+\mu dz') - f(z_1, z'_1)}{\mu dz'}.$$

By hypothesis e_1 and e_2 go to zero when λ and μ are made to go to zero in any manner whatever and since the second and third terms of e_1 and e_2 cancel in the limit it follows that

$$(5.6)\quad \lim_{\lambda,\mu\to 0} \tau\phi(\lambda,\mu)/\lambda = \lim_{\lambda,\mu\to 0} \sigma\phi(\lambda,\mu)/\mu = 0.$$

Since this is true for every choice of τ and σ one concludes that (5.2 b) holds. This proves the necessity of the condition.

Suppose that as $\lambda, \mu \to 0$, $\tau \to 0$ and $\sigma \to 1$; this would make $\phi(\lambda,\mu)/\mu \to 0$ while an interchange of τ and σ makes $\phi(\lambda,\mu)/\lambda \to 0$. The disturbing case is when $\tau \to \infty$, $\sigma \to -\infty$ say, while $\tau+\sigma \to 1$ with $\lambda, \mu \to 0$. Since (5.6) has to be satisfied in this case it follows that $1/\tau$ and $1/\sigma$ are infinitesimal functions of λ and μ of a lower order than $\phi(\lambda,\mu)/\lambda$ and $\phi(\lambda,\mu)/\mu$ respectively. This result will be used in the sufficiency proof.

In the sufficiency proof we may take

$$\partial f/\partial z_1 = A \quad \text{and} \quad \partial f/\partial z'_1 = B$$

since by hypothesis these partial derivatives exist and are finite. This permits us to write (5.1) in form (5.4) from which we may again read off e_1 and e_2 as given in (5.5). The hypothesis (5.2 b) together with the restrictions imposed on τ and σ in the last paragraph, evidently imply the validity of (5.6). This makes the first term in each of e_1 and e_2 zero in the limit while the second terms in each cancel in the limit as before. From the hypotheses we have thus constructed (5.1) wherein e_1 and e_2 have the prescribed properties.

6. Thomae's test. The usual formulation of this test is as follows: The existence and finiteness of $\partial f/\partial z_1$, $\partial f/\partial z'_1$ together with the continuity of one, say $\partial x/\partial z$, implies regularity. Evidently continuity of a partial derivative is unnecessary for R-regularity; while continuity of just one of the existing partial derivatives is not sufficient for Y-regularity. For if it were, the secondary conditions would be satisfied. We find that

$$\lim_{\lambda,\mu\to 0}\frac{\phi(\lambda,\mu)}{\lambda dz}=\lim_{\lambda,\mu\to 0}\frac{f(z_1+\lambda dz,z'_1+\mu dz')-f(z_1,z'_1+\mu dz')}{\lambda dz}$$
$$-\frac{f(z_1+\lambda dz,z'_1)-f(z_1,z'_1)}{\lambda dz}$$
$$=\lim_{\lambda\to 0}\frac{\partial f(z_1,z'_1+\mu dz')}{\partial z_1}-\frac{\partial f}{\partial z_1}$$

which vanishes only if $\partial f/\partial z_1$ is continuous in z'. In a similar way one finds that

$$\lim_{\lambda,\mu\to 0}\frac{\phi(\lambda,\mu)}{\mu dz'}=\lim_{\lambda\to 0}\frac{\partial f(z_1+\lambda dz,z'_1)}{\partial z'_1}-\frac{\partial f}{\partial z'_1}$$

vanishes if $\partial f/\partial z'_1$ is continuous in z.

Continuity of one partial derivative seems to be of little value as a test for Y-regularity. The continuity of one must be supplemented by some other condition possibly by the continuity of the other.

7. *R*-*Y*-regularity relations. It should be expected that Y-regularity implies R-regularity. That the latter is a special case of the former is seen by replacing μ by λ in $\phi(\lambda,\mu)$ as defined in (5. 3). We find that

$$(7.1)\qquad \frac{\phi(\lambda,\mu)}{\lambda dz}=\frac{f(z_1+\lambda dz,z'_1+\lambda dz')-f(z_1,z'_1+\lambda dz')}{\lambda dz}$$
$$-\frac{f(z_1+\lambda dz,z'_1)-f(z_1,z'_1)}{\lambda dz}$$

which becomes zero in the limit. Thus the secondary conditions (5. 2 b) are seen to be identically satisfied.

The question naturally arises: What less restrictive condition on the definition for Y-regularity is sufficient for an R-regular function to be at the same time a Y-regular function. A sufficient restriction is that λ and μ be infinitesimals of the same order. In fact when $k\lambda$ is substituted for μ in $\phi(\lambda,\mu)$ (7. 1) will remain unchanged, since dz' will merely carry the additional factor k which may be absorbed by it.

By imposing a restriction on the function rather than on the definition for Y-regularity we have

THEOREM 3. *For an R-regular function $f(z, z')$ to be at the same time a Y-regular function it is sufficient for $\partial f/\partial z$ to be continuous in z' and $\partial f/\partial z'$ to be continuous in z at P_1.*

That this theorem is true follows from the discussion of the Thomae test. In fact it was seen there that when the conditions of the theorem are applied to $f(z, z')$ the secondary conditions (5. 2 b) are identically satisfied.

8. The geometry. The special class of polygenic functions considered here are obtained from the function $f(z, z')$ by making $z' = \bar{z}$ the conjugate of z. We first superimpose the z' complex plane on the z plane making the origins and corresponding axes incident. If we now replace z' by $\bar{z}$ in $f(z, z')$ this function becomes the polygenic function $f(z)$ in the sense that its z derivative at a point depends on the direction of approach to the point.

9. *R*-regularity of $f(z)$. The replacement of z' by $\bar{z}$ changes (4. 2) into

$$f(z) = f(z_1) + \frac{\partial f}{\partial z_1}(z - z_1) + \frac{\partial f}{\partial \bar{z}_1}(\bar{z} - \bar{z}_1) + \eta_1 \tag{9.1}$$

where $\partial f/\partial z_1$ and $\partial f/\partial \bar{z}_1$ are written for $\partial f(z_1, z'_1)/\partial z_1$ and $\partial f(z_1, z'_1)/\partial z'_1$ respectively wherein z'_1 is replaced by $\bar{z}_1$.

THEOREM 4. *The necessary and sufficient conditions that the polygenic function $f(z)$ be R-regular at a point are the existence and finiteness of the areal and mean derivatives at the point.*[6]

In the proof of this theorem we note that (4. 2) furnished the necessary and sufficient condition for R-regularity of $f(z, z')$. Thus (9. 1), a consequence of (4. 2), furnishes the necessary and sufficient condition for R-regularity of $f(z)$. Also since the existence and finiteness of $\partial f/\partial z_1$ and $\partial f/\partial z'_1$ furnished a second set of necessary and sufficient conditions for R-regularity of $f(z, z')$ all that is left to do is to prove that $\partial f/\partial z_1$ and $\partial f/\partial \bar{z}_1$ are the areal and mean derivatives respectively.

By definition the areal derivative

$$\frac{\partial f}{\partial \alpha} = \lim_{\sigma \to 0} \frac{1}{2i\sigma} \int_C f(z)\,dz = \lim_{r \to 0} \frac{1}{2\pi r^2 i} \int_0 f(z)\,dz$$

[6] V. C. Poor, *Transactions of the American Mathematical Society*, vol. 32 (1930), p. 216.

where σ is chosen as the area of the circle radius r center at $z = z_0$ while the integral is to be taken around the circle. Briefly then we put $z - z_1 = re^{i\theta}$ and replace $f(z)$ by the right member of (9. 1); we find that

$$\frac{\partial f}{\partial \alpha} = \lim_{r\to 0} \frac{1}{2\pi r^2 i} \frac{\partial f}{\partial z_1} \int_0 (\bar{z} - \bar{z}_1)\, dz = \lim_{r\to 0} \frac{1}{2\pi r^2 i} \frac{\partial f}{\partial \bar{z}_1} \int_0 2\pi r^2 i' d\theta = \frac{\partial x}{\partial \bar{z}_1}.$$

All other integrals vanish or become zero in the limit. In exactly the same way we show that

$$\frac{\partial f}{\partial \beta} = \lim_{r\to 0} \frac{-1}{2\pi i' r^2} \int_0 f(z) d\bar{z} = \frac{\partial f}{\partial z_1}.$$

10. *Y*-regularity of $f(z, z')$.

THEOREM 5. *The necessary and sufficient conditions for the existence of a Y-differential of the polygenic function $f(z)$ at $z = z_1$ are*

(a) *the existence and finiteness of the areal and mean derivatives*

$$\partial f/\partial \alpha \quad \text{and} \quad \partial f/\partial \beta$$

and

(b) $$\lim_{\lambda,\mu\to 0} \frac{\phi(\lambda, \mu)}{\lambda} = 0 = \lim_{\lambda,\mu\to 0} \frac{\phi(\lambda, \mu)}{\mu}.$$

This should be evident from the geometry. However, there is a lack of symmetry due to the μ. Putting $z - z_1 = \lambda dz$ and therefore $\bar{z} - \bar{z}_1 = \lambda d\bar{z}$ into $f(z_1 + \lambda dz, \bar{z}_1 + \mu d\bar{z})$ we find that

$$f[z, \bar{z}_1 + (\mu/\lambda)(\bar{z} - \bar{z}_1)]$$
$$= f(z_1, \bar{z}_1) + [A + e_1(\lambda, \mu)](z - z_1) + [B + e_2(\lambda, \mu)](\mu/\lambda)(\bar{z} - \bar{z}_1).$$

This however may be used for $f(z)$ since as r approaches zero this approaches $f(z_1)$ as a limit. We may thus use this for $f(z)$ in determining the areal and mean derivatives so that $\partial f/\partial \alpha = B$ and $\partial f/\partial \beta = A$ as before.

Also $e_1(\lambda, \mu)$ and $e_2(\lambda, \mu)$ may be constructed as previously, noting that $\partial f/\partial \bar{z}_1$ and $\partial f/\partial z_1$ which are defined above are the areal and mean derivatives respectively. Further argument follows that in Theorem 2. This should be conclusive without further detail.

II. THE SUGGESTED GENERALIZATIONS.

11. The line integral. We have seen that the regular function $f(z, z')$ is analytic in z and z' in their separate domains. The line integral of this function from z_0 to z_1 is thus independent of the path of integration.

We define $F(z_1, z'_1)$ by the equation

$$F(z_1, z'_1) = \int_{z_0}^{z_1} f(z, z'_1)\, dz \tag{11.1}$$

where the integral is to be taken along a rectifiable path in the domain D.

THEOREM 6. *If $f(z, z')$ is Y-regular at (z_1, z'_1) its integral $F(z_1, z'_1)$ has a double incrementary ratio which tends to the limit $\partial f(z_1, z'_1)/\partial z'_1$ or $\partial f(z, z')/\partial z'$ at (z_1, z'_1) that is*

$$\begin{aligned}
&\lim_{\lambda,\mu\to 0} \frac{\Phi(\lambda, \mu)}{\lambda dz \mu dz'} \\
&= \lim_{\lambda,\mu\to 0} \frac{F(z_1 + \lambda dz, z'_1 + \mu dz') - F(z_1 + \lambda dz, z'_1) - F(z_1, z'_1 + \mu dz') + F(z_1, z'_1)}{\lambda dz \mu dz'} \\
&= \frac{\partial f(z_1, z'_1)}{\partial z'_1}.
\end{aligned}$$

Expressed in terms of the integral

$$\begin{aligned}
\Phi(\lambda, \mu) &= \int_{z_0}^{z_1 + \lambda dz} f(z, z'_1 + \mu dz')\, dz - \int_{z_0}^{z_1 + \lambda dz} f(z, z'_1)\, dz \\
&\quad - \int_{z_0}^{z_1} [f(z, z'_1 + \mu dz') - f(z, z'_1)]\, dz \\
&= \int_{z_1}^{z_1 + \lambda dz} [f(z, z'_1 + \mu dz') - f(z, z'_1)]\, dz.
\end{aligned} \tag{11.2}$$

If this integral is transformed by the substitution $z = z_1 + \lambda s$ one finds that

$$\begin{aligned}
&\lim_{\lambda,\mu\to 0} \frac{\Phi(\lambda, \mu)}{\lambda dz \mu dz'} \\
&\quad = \lim_{\lambda,\mu\to 0} \frac{1}{dz} \int_0^{dz} \frac{f(z_1 + \lambda s, z'_1 + \mu dz') - f(z_1 + \lambda dz, z'_1)}{\mu dz'}\, ds \\
&\quad = \lim_{\lambda,\mu\to 0} \frac{1}{dz} \int_0^{dz} \frac{\partial f(z_1, z'_1)}{\partial z'_1}\, ds = \frac{\partial f(z_1, z'_1)}{\partial z'_1}.
\end{aligned}$$

However if μ is made to go to zero first the limit is $\partial f(z_1 + \lambda s, z'_1)/\partial z'_1$ which becomes $\partial f(z_1, z'_1)/\partial z'_1$ if $\partial f/\partial z'_1$ is continuous at (z_1, z'_1).

We may observe, as a corollary, that if $f(z, z')$ is R-regular at (z_1, z'_1) the theorem is valid without the continuity condition since, in this case $\mu = k\lambda$. Also, since the above result is true at any point z_1, z'_1 in the domain D and its conjugate $D' = \bar{D}$, z'_1 may be replaced by $\bar{z}_1$ the conjugate to z. But we have seen that $\partial f(z_1, \bar{z}_1)/\partial \bar{z}_1$ is the areal derivative of $f(z)$ at $z = z_1$. This

result, of course, does not imply that the integrand in $F(z, z')$ is $f(z)$; the integral would in that case depend on the path. A similar result may be obtained by taking the path of integration from z'_0 to z'_1 in the conjugate domain D' with z' as the variable of integration, leading finally to the mean derivative in the limit.

THEOREM 7. *If the integral* $\int_{z_0}^{z_1} f(z, z'_1)dz$ *is independent of the path and if* $f(z, z')$ *is Y-regular not only at* (z_1, z'_1) *but also at every point of the path of integration from* z_0 *to* z_1 *then* $\partial f/\partial z'_1$ *is an integrable function of* z *and* $\partial F/\partial z'_1$ *exists and is given by differentiation under the integral sign.*

In the proof we write the previous theorem

$$\lim_{\lambda,\mu\to 0} \frac{1}{\lambda dz \mu dz'} \int_{z_1}^{z_1+\lambda dz} [f(z, z'_1 + \mu dz') - f(z, z'_1)]dz = \frac{\partial f(z_1, z'_1)}{\partial z'_1}.$$

Hence

$$\lim_{\lambda\to 0} \frac{1}{\lambda dz} \int_{z_1}^{z_1+\lambda dz} [f(z, z'_1 + \mu dz') - f(z, z'_1)]dz = \frac{\partial f(z, z'_1)}{\partial z'_1} \mu dz' + e(\mu)\mu dz'$$

where $e(\mu)$ is an infinitesimal function of μ. But according to Lebesgue's theorem [7] the left member is equal to the integrand at $z = z_1$. Thus

$$\frac{1}{\mu dz'} [f(z, z'_1 + \mu dz') - f(z, z'_1)] = \frac{\partial f(z, z'_1)}{\partial z'_1} + e(\mu)$$

where we have written z for z_1. When this result is multiplied by dz and integrated along the rectifiable path one finds that

$$\frac{1}{\mu dz'} \int_{z_0}^{z_1} [f(z, z'_1 + \mu dz' - f(z, z'_1)]dz = \int_{z_0}^{z_1} \frac{\partial f(z, z'_1)}{\partial z'_1} dz + \int_{z_0}^{z_1} edz.$$

When the left member is identified through the definition (11.1) of $F(z_1, z'_1)$ we find that

$$\frac{F(z_1, z_1 + \mu dz') - F(z_1, z'_1)}{\mu dz'} = \int_{z_0}^{z_1} \frac{\partial f(z, z'_1)}{\partial z'_1} dz + \int_{z_0}^{z_1} edz.$$

In making μ go to zero and noting that the second integral goes to zero with μ we find that

$$\frac{\partial F(z_1, z'_1)}{\partial z'_1} = \int_{z_0}^{z_1} \frac{\partial f(z, z'_1)}{\partial z'_1} dz.$$

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[7] Lebesgue, *Annales de l'École Normale*, vol. 27, pp. 363-387.

THEOREM 8. *If $f(z, z')$ is Y-regular then $F(z_1, z'_1)$ is Y-regular.*

We have already seen that

$$\frac{\partial F}{\partial z'_1} = \int_{z_0}^{z_1} \frac{\partial f(z, z'_1)}{\partial z'_1}\, dz$$

while

$$\partial F/\partial z_1 = f(z_1, z'_1).$$

Thus the primary conditions (5. 2 a) for Y-regularity are satisfied.

For the secondary conditions we have by (11. 2)

$$\frac{\Phi(\lambda, \mu)}{\lambda dz} = \frac{1}{\lambda dz} \int_{z_1}^{z_1+\lambda dz} [f(z, z'_1 + \mu dz') - f(z, z')]dz.$$

Here, if μ goes to zero first the integrand becomes identically zero so that

$$\operatorname*{limit}_{\lambda,\mu\to 0} \frac{\Phi(\lambda, \mu)}{\lambda} = 0.$$

If λ goes to zero first one has

$$\operatorname*{limit}_{\lambda,\mu\to 0} \frac{\Phi(\lambda, \mu)}{\lambda} = \operatorname*{limit}_{\mu\to 0} f(z_1, z'_1 + \mu dz') - f(z_1, z'_1) = 0.$$

Further

$$\frac{\Phi(\lambda, \mu)}{\mu dz'} = \frac{1}{\mu dz'} \int_{z_0}^{z_1} [f(z, z'_1 + \mu dz') - f(z, z'_1)]dz;$$

when λ goes to zero first here, the integral vanishes while if μ goes to zero first

$$\operatorname*{limit}_{\lambda,\mu\to 0} \frac{\Phi(\lambda, \mu)}{\mu dz'} = \operatorname*{limit}_{\lambda\to 0} \int_{z_1}^{z_1+\lambda dz} \frac{\partial f(z_1, z'_1)}{\partial z'_1}\, dz = 0.$$

Thus in all cases the secondary conditions (5. 2 b) are satisfied.

THEOREM 9. *If $f(z, z')$ and $g(z, z')$ are each Y-regular their product is Y-regular.*

If this theorem were not true the utility of the definition would be in question. Its proof will be left to the reader with the suggestion that $\phi(f)$ and $\phi(g)$ be constructed. Transpose $f(P)$ and $g(P)$, $P \equiv (z_1 + \lambda dz, z'_1 + \mu dz')$ and form the product. From this $\phi(fg)$ may be constructed.

III. TAYLOR SERIES EXPANSION OF A POLYGENIC FUNCTION.

12. Taylor series for $f(z, z')$. Let $f(z, z')$ be R-regular at every point of a neighborhood of $z = z_1$ and of a neighborhood of $z' = z'_1$. Let C and C' be the bounding curves, without double points, of the respective neighborhoods and so chosen that C does not contain the point z'_1 nor C' the point z_1 in their bounded areas. Then it is well known that the Cauchy second law generalizes into [8]

$$f(z, z') = -\frac{1}{4\pi^2}\int_C dt \int_{C'} \frac{f(t, t')\,dt'}{(t-z)(t'-z')} \tag{12.1}$$

where as usual t and t' are points on C and C' respectively. From this it will be evident that

$$\frac{\partial^{m+n} f(z, z')}{\partial z^m \partial z'^n} = -\frac{m!\,n!}{4\pi^2}\int_C dt \int_{C'} \frac{f(t, t')\,dt'}{(t-z)^{m+1}(t'-z')^{n+1}}. \tag{12.2}$$

We now restrict the contours C and C' to be circles, with centers at z_1 and z'_1 and radii R and R' respectively, and so chosen that neither circle contains the center of the other in its interior or on its boundary. Further

$$\begin{aligned}\frac{1}{(t-z)(t'-z')} &= \frac{1}{t-z_1-(z-z_1)} \cdot \frac{1}{(t'-z'_1)-(z'-z'_1)} \\ &= \frac{1}{(t-z_1)(t'-z'_1)} \cdot \frac{1}{1-\dfrac{z-z_1}{t-z_1}} \cdot \frac{1}{1-\dfrac{z'-z'_1}{t'-z'_1}}.\end{aligned}$$

may be expanded into a power series by simply performing the indicated division in the last two factors and multiplying the resulting series. One finds that

$$\frac{1}{(t-z')(t'-z')} = \frac{1}{(t-z_1)(t'-z'_1)} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \left(\frac{z-z_1}{t-z_1}\right)^m \left(\frac{z'-z'_1}{t'-z'_1}\right)^n. \tag{12.3}$$

If now two other concentric circles radii r and r' are so chosen that

$$|z-z_1| < r < R \quad \text{and} \quad |z'-z'_1| < r' < R'$$

[8] Goursat, *Cours d'Analyse*, vol. II, p. 272.

then the series (12. 3) will be uniformly convergent, for

$$\left| \left(\frac{z-z_1}{t-z_1}\right)^m \left(\frac{z'-z'_1}{t'-z'_1}\right)^n \right| \leq \frac{1}{RR'} \left(\frac{r}{R}\right)^m \left(\frac{r'}{R'}\right)^n.$$

From (12. 1) and (12. 3) we find that

$$f(z,z') = -\frac{1}{4\pi^2} \int_C dt \int_{C'} \frac{f(t,t')dt'}{(t-z)(t'-z')}$$

$$= -\frac{1}{4\pi^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (z-z_1)^m (z'-z'_1)^n \int_C dt \int_{C'} \frac{f(t,t')dt'}{(t-z_1)^{m+1}(t'-z'_1)^{n+1}}$$

and applying (12. 2) to the integral we find that

$$(12.4) \quad f(z,z') = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m!\,n!} \cdot \frac{\partial^{m+n} f(z_1, z'_1)}{\partial z_1^m \partial z'^n_1} (z-z_1)^m (z'-z'_1)^n .$$

where, by definition, zero factorial is 1 and the zero derivative of the function is the function itself.

If the maximum value of $|f(z,z')|$ on the circles radii r and r' is M then

$$(12.5) \quad \left| \frac{1}{m!\,n!} \frac{\partial^{m+n} f(z,z')}{\partial z_1^m \partial z'^n_1} \right| < \frac{1}{4\pi^2} \frac{M}{R^{m+1} R'^{n+1}} 2\pi R \cdot 2\pi R' . = \frac{M}{R^m R'^n}$$

which is obtained by applying (12. 2) to the integral form of the coefficient. Thus our series (12. 4) is convergent.

13. The Taylor expansion of the polygenic function. We have seen that when $f(z,z')$ is R-regular $f(z,z')$ is given by equation (4. 2). Also when z' is replaced by $\bar{z}$ this becomes

$$f(z) = f(z_1) + \frac{\partial f}{\partial z_1}(z-z_1) + \frac{\partial f}{\partial \bar{z}_1}(\bar{z}-\bar{z}_1) + \eta_1.$$

$\partial f/\partial z_1$ and $\partial f/\partial \bar{z}_1$ are, respectively, $\partial f(z_1,z'_1)/\partial z_1$ and $\partial f(z_1,z'_1)/\partial z'_1$ wherein z'_1 has been replaced by $\bar{z}_1$. It has also been shown that $\partial f/\partial z_1 = \partial f/\partial \beta$, $\partial f/\partial \bar{z}_1 = \partial f/\partial \alpha$ at $z = z_1$ are the areal and mean derivatives respectively. Since all derivatives of $f(z,z')$ are regular it follows that all derivatives of $f(z)$ will be areal and mean derivatives. Hence

$$\partial^{m+n} f/\partial z_1^m \partial \bar{z}_1^n = \partial^{m+n} f/\partial \beta^m \partial \alpha^n.$$

A substitution of this result into (12.4) makes

$$(13.1) \qquad f(z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m!\,n!} \frac{\partial^{m+n} f(z_1)}{\partial \beta^m \partial \alpha^n} (z - z_1)^m (\bar{z} - \bar{z}_1)^n$$

which is the required expansion of $f(z)$ into a Taylor Series.

(a) The Taylor expansion for $f(z, z')$ includes the case when the domain D' is identified with the conjugate domain $\bar{D}$ of D, making $z'_1 = \bar{z}_1$ the center of the circle C'_1.

(b) When z' is replaced by $\bar{z}$ (12.1) is no longer valid, i. e. the polygenic function $f(z)$ cannot be expressed in any such form.

(c) The derivative $\partial f(z_1, z'_1)/\partial z'_1$ can be obtained from the function itself and from the double integral in (12.1), but $\partial f/\partial \bar{z}_1$ can be obtained only from $\partial f(z_1, z'_1)/\partial z'_1$ by putting $z'_1 = \bar{z}_1$. A simliar situation arises for $\partial f/\partial z_1$, of course.

(d) We obtain the right member of (12.5) from the right member of (12.2) while the left member may be obtained from the function $f(z_1, z'_1)$ itself.

(e) The point z'_1 is by our geometric set up the very same point as $\bar{z}_1$. Thus the left member of (12.5) will retain the same value when z'_1 is replaced by $\bar{z}_1$, and therefore the inequality persists. But in this case the left member of (12.5) becomes a combination of areal and mean derivatives as indicated; The convergence of the series is thus established.

(f) This expansion of a polygenic function into a Taylor series is at least a partial solution of this classical problem. The method used should be of some future value.

THE UNIVERSITY OF MICHIGAN.

CONJUGAL QUADRICS AND THE QUADRIC OF MOUTARD.*

By M. L. MacQueen.

1. Introduction. Among the three-parameter family of quadrics having contact of the second order with a surface at a point, Grove [1; p. 231] has introduced a two-parameter family of quadrics, called *conjugal quadrics*, which are associated with a given conjugate net on the surface. They are found to include certain pencils of quadrics which are of interest, namely, the quadrics of Darboux, the quadrics which have third-order contact [2; p. 421] with both curves of a conjugate net at a point, and the quadrics of Davis [3; p. 12].

The conjugal quadrics which are associated with the curves of a conjugate net at a point P_x of a surface and the quadric of Moutard in the direction λ of a tangent of a curve of the net intersect in the asymptotic tangents through the point P_x and in a conic which lies in a plane through P_x. It is the purpose of this paper to study the envelope of this plane when λ varies.

In **2**, we are concerned with the calculation of the equations of the envelope, which is found to be a cone of the sixth class with its vertex at the point P_x. A study of the envelope and its polars, in **3**, enables us to obtain characterizations of certain conjugal quadrics which appear to be of some interest. Among other things, we present several geometric definitions of the general canonical line of the first kind. The last four sections, which are devoted to a brief study of special cases, contain geometric characterizations of unique conjugal quadrics and new interpretations of particular canonical lines.

In a recent paper, Chang [4; p. 926] studied the envelope of the plane containing the residual conic of intersection of the quadrics of Darboux and the quadric of Moutard for a direction λ at a point P_x when λ varies. Consequently, it is of interest to obtain some of his results as a special case of the problem considered here.

2. The envelope. Let the differential equations of a non-ruled surface S in ordinary projective space be written in the Fubini canonical form [5; p. 69]

$$(2.1) \qquad x_{uu} = px + \theta_u x_u + \beta x_v, \qquad x_{vv} = qx + \gamma x_u + \theta_v x_v \qquad (\theta = \log \beta\gamma).$$

* Received March 20, 1945.

We select an ordinary point P_x of the surface S as one vertex of the usual local tetrahedron of reference x, x_u, x_v, x_{uv}. A conjugate net N_λ on the surface S can be represented by a curvilinear differential equation of the form

$$dv^2 - \lambda^2 du^2 = 0 \qquad (\lambda \neq 0), \tag{2.2}$$

where λ is a function of u, v. The two curves of the net N_λ that pass through the point P_x may be denoted by C_λ and $C_{-\lambda}$ according as the direction dv/du has the value λ or $-\lambda$.

With the introduction of nonhomogeneous projective coördinates x, y, z defined in the customary way, the equation of the conjugal quadrics associated with the net N_λ at the point P_x can be written in the form

$$xy - z + h(\gamma\lambda^2 xz + \beta yz/\lambda^2) + lz^2 = 0, \tag{2.3}$$

in which h, l are arbitrary parameters.

Among the conjugal quadrics, perhaps the most important are those for which h is a constant. Such quadrics include the quadrics of Darboux with $h = 0$, the quadrics of Davis with $h = 1$, the quadrics with $h = -\frac{1}{3}$ which have third-order contact with both curves of the net N_λ, and the quadrics corresponding to various constant values of h for which Grove has given geometric characterizations [1; p. 233].

We shall suppose from now on that h is a constant which is independent of u and v.

The quadric of Moutard for a direction λ at the point P_x of the surface has the equation

$$\begin{aligned} &36\lambda^3(xy - z) + 12\lambda^2(\gamma\lambda^3 - 2\beta)xz - 12\lambda(2\gamma\lambda^3 - \beta)yz \\ &\quad + [3\beta\phi\lambda - 12\beta\psi\lambda^2 - 18\theta_{uv}\lambda^3 - 12\gamma\phi\lambda^4 + 3\gamma\psi\lambda^5 - 4(\beta + \gamma\lambda^3)^2]z^2 = 0, \end{aligned} \tag{2.4}$$

where the functions ϕ and ψ are defined by

$$\phi = (\log \beta\gamma^2)_u, \qquad \psi = (\log \beta^2\gamma)_v.$$

The two quadrics (2.3) and (2.4) intersect in a curve consisting of the asymptotic tangents at the point P_x of the surface, and in a conic which lies in the plane π_λ whose equation is

$$\begin{aligned} &12\lambda^2(2\beta - \gamma\lambda^3 + 3h\gamma\lambda^3)x + 12\lambda(3h\beta - \beta + 2\gamma\lambda^3)y \\ &\quad + [4\beta^2 - 3\beta\phi\lambda + 12\beta\psi\lambda^2 + (8\beta\gamma + 36L\beta\gamma)\lambda^3 + 12\gamma\phi\lambda^4 \\ &\quad - 3\gamma\psi\lambda^5 + 4\gamma^2\lambda^6]z = 0, \end{aligned} \tag{2.5}$$

in which we have placed

$$l = L\beta\gamma - \tfrac{1}{2}\theta_{uv}. \tag{2.6}$$

The homogeneous plane coördinates of the plane π_λ are given by the equations

$$
\begin{aligned}
&\rho u_1 = 0,\\
&\rho u_2 = 24\beta\lambda^2 - 12(1-3h)\gamma\lambda^5,\\
&\rho u_3 = -12(1-3h)\beta\lambda + 24\gamma\lambda^4,\\
&\rho u_4 = 4\beta^2 - 3\beta\phi\lambda + 12\beta\psi\lambda^2 + (8\beta\gamma + 36L\beta\gamma)\lambda^3 + 12\gamma\phi\lambda^4\\
&\qquad - 3\gamma\psi\lambda^5 + 4\gamma^2\lambda^6,
\end{aligned}
\tag{2.7}
$$

where ρ is a proportionality factor not zero.

The envelope of the plane π_λ when λ varies may be found by homogeneous elimination of λ from equations (2.7). For convenience, we follow Chang [4; p. 927] and obtain, after some calculation based on equations (2.7), an equivalent system of equations, namely,

$$
\begin{aligned}
&\pi\lambda^3 + \omega\lambda^2 + t\lambda + s = 0,\\
&\sigma\lambda^3 + \tau\lambda^2 + \omega\lambda + p = 0,
\end{aligned}
\tag{2.8}
$$

whose coefficients are given by the following formulas which preserve Chang's notation:

$$
\begin{aligned}
\pi &= -\frac{2(1+3h)(1-h)}{1-3h}\,\gamma u_2 + (1-6h)\psi u_3,\\
\omega &= \tfrac{1}{2}(7+3h)\psi u_2 + \tfrac{1}{2}(7+3h)\phi u_3 - 6(1+3h)(1-h)u_4,\\
t &= (1-6h)\phi u_2 + 6\left[\frac{(1-h)^2 + 2(1-3h)L}{1-3h}\right]\beta u_3,\\
s &= [4(1-h) + 6(1-3h)L]\beta u_2,\\
\sigma &= [4(1-h) + 6(1-3h)L]\gamma u_3,\\
\tau &= (1-6h)\psi u_3 + 6\left[\frac{(1-h)^2 + 2(1-3h)L}{(1-3h)}\right]\gamma u_2,\\
p &= (1-6h)\phi u_2 - \frac{2(1+3h)(1-h)}{1-3h}\,\beta u_3,
\end{aligned}
\tag{2.9}
$$

in which we have assumed $h \neq 1$ and $h \neq \pm\frac{1}{3}$. These excluded cases will be considered in the last three sections.

If we eliminate λ from equations (2.8), we find that *the envelope of the plane π_λ is the cone whose equations in local plane coördinates are $u_1 = 0$ and*

$$
\begin{vmatrix}
\pi & \omega & t & s\\
\sigma & \tau & \omega & p\\
0 & p\pi - \sigma s & \omega p - \tau s & pt - \omega s\\
\omega\sigma - \pi\tau & \sigma t - \omega\pi & \sigma s - \pi p & 0
\end{vmatrix} = 0.
\tag{2.10}
$$

In the first place, if L has the value given by

$$
L = -\frac{2}{3}\left(\frac{1-h}{1-3h}\right),
\tag{2.11}
$$

equations (2.9) show that $t = p$, $\tau = \pi$, and $s = \sigma = 0$, in consequence of which it is not difficult to verify that the determinant appearing in equation (2.10) is identically zero. From this fact we infer that the envelope of the plane π_λ differs from the cone (2.10), in case L is defined by equation (2.11). Consequently we shall assume, for the present, that the parameter L is arbitrary and is not given by (2.11).

3. Properties of the envelope. The third polar of the tangent plane $z = 0$ of the surface at the point P_x with respect to the cone (2.10) is a cone of class three which is found to have the equations $u_1 = 0$ and

$$\begin{aligned} 6u_2u_3\Big[u_4 &- \frac{(2-4h-3h^2)}{4(1+3h)(1-3h)(1-h)}\phi u_3 \\ &- \frac{(2-4h-3h^2)}{4(1+3h)(1-3h)(1-h)}\psi u_2\Big] \\ &- \frac{1}{(1-3h)^2}(\beta u_3^3 + \gamma u_2^3) = 0. \end{aligned} \tag{3.1}$$

It will be observed that this equation is independent of L, so that *all of the cones* (2.10) *with different values of the parameter* L *have in common the same third polar with respect to the tangent plane* $z = 0$.

The cusp-axis of the cone (3.1) is the canonical line $l_1(k)$ for which

$$k = -\frac{(2-4h-3h^2}{4(1+3h)(1-3h)(1-h)}. \tag{3.2}$$

Thus we have a construction which yields the general canonical line of the first kind and which may be described in the following theorem:

If the parameter L *is not defined by* (2.11), *corresponding to each conjugal quadric* (2.3), *except those with* $h = 1$ *and* $h = \pm\frac{1}{3}$, *there is a canonical line* $l_1(k)$ *for which* k *is defined by the formula* (3.2), *and this line is the cusp-axis of the cone* (3.1).

In particular, it is perhaps worthy of remark that the conjugal quadrics with $h = 0$, $h = 1/6$, and $h = 2/3$ all correspond in this manner to the first directrix of Wilczynski.

The second polar of the tangent plane $z = 0$ with respect to the cone (2.10) is a cone of the fourth class with the equations $u_1 = 0$ and

$$\begin{aligned} 6u_2u_3\omega^2 - 6\Big[&\frac{(1-6h)}{(1-3h)}(\psi u_2 + \phi u_3)u_2u_3 \\ &- \frac{2(1+3h)(1-h)}{(1-3h)^2}(\beta u_3^3 + \gamma u_2^3)\Big]\omega + [\ldots]u_2u_3 = 0, \end{aligned} \tag{3.3}$$

where $[\ldots]$ is independent of ω but does contain the parameter L.

The equation of any plane through the v-tangent $x = z = 0$, distinct from the tangent plane, can be written in the form

$$x + az = 0, \tag{3.4}$$

in which a is a parameter. If the homogeneous plane coördinates of this plane are substituted in equation (3.3), we find that the plane (3.4) is tangent to the cone (3.3) if, and only if,

$$a = \frac{7 + 3h}{12(1 + 3h)(1 - h)}\psi. \tag{3.5}$$

Similarly, a plane through the u-tangent $y = z = 0$ has the equation

$$y + bz = 0,$$

in which b is a parameter, and is tangent to the cone (3.3) in case

$$b = \frac{7 + 3h}{12(1 + 3h)(1 - h)}\phi. \tag{3.6}$$

Thus we prove the following theorem:

Through each asymptotic tangent at the point P_x of the surface S there is a plane which is tangent to all of the cones (3.3) *with variable L. These two planes intersect in the canonical line $l_1(k)$ for which*

$$k = -\frac{7 + 3h}{12(1 + 3h)(1 - h)}. \tag{3.7}$$

The equation of any plane, except the tangent plane $z = 0$, through the tangent line in the direction λ at the point P_x of the surface is

$$y - \lambda x = \mu z, \tag{3.8}$$

where μ is a parameter. If the plane coördinates of this plane are substituted in equation (3.3), we find

$$A\mu^2 + B\mu + C = 0, \tag{3.9}$$

in which

$$\begin{aligned} A &= 12(1 + 3h)(1 - h)\lambda, \\ B &= \frac{3(2 - 4k - 3h^2)}{1 - 3h}(\phi - \psi\lambda)\lambda - \frac{2(1 + 3h)(1 - h)}{(1 - 3h)^2}(\beta - \gamma\lambda^3), \end{aligned} \tag{3.10}$$

the definition of C not being essential to our work. Hence, through the tangent line in the direction λ there are two planes p_1, p_2, besides the tangent plane of the surface, which are tangent to the cone (3.3). The harmonic conjugate of the tangent plane $z=0$ with respect to the two planes p_1, p_2 is the plane p^* whose coördinates are given by

$$(3.11) \qquad \rho u_1 = 0, \qquad \rho u_2 = -A\lambda, \qquad \rho u_3 = A, \qquad \rho u_4 = B.$$

The plane p^* is tangent to the cone (3.1) if, and only if,

$$(3.12) \qquad 2h(1+3h)(1-h)(\beta-\gamma\lambda^3) - 3h(1-3h)(2-4h-3h^2)(\phi-\psi\lambda)\lambda = 0.$$

This condition is obviously satisfied if $h=0$, that is, if the conjugal quadric (2.3) is a quadric of Darboux. If $h \neq 0$, equation (3.12) represents the triple of directions D^*_k which are conjugate to the D_k directions introduced, respectively, by Wilkins [6; p. 177] and Bell [7; p. 787]. Moreover, condition (3.12) is satisfied if $\phi - \psi\lambda = 0$, $\beta - \gamma\lambda^3 = 0$, so that the curve C_λ is a curve of Segre which is tangent to the first canonical tangent at P_x. These results may be summarized in the following statement:

There are two planes p_1, p_2, besides the tangent plane of the surface S, through the tangent line in the direction λ at the point P_x which are tangent to the cone (3.3). *The plane p^* which is the harmonic conjugate of the tangent plane $z=0$ with respect to the planes p_1 and p_2 envelopes the cone* (3.1) *in case any one of the following conditions holds:*

(i) *the conjugal quadric* (2.3) *is a quadric of Darboux;*

(ii) *the direction of the curve C_λ is a direction D^*_k;*

(iii) *the curve C_λ is a curve of Segre which is tangent to the first canonical tangent at P_x.*

Moreover, it is not difficult to verify the truth of the following theorem:

At a point P_x of the surface S, let p_1, p_2 denote the two planes through the tangent line in the direction λ at the point P_x which are tangent to the cone (3.3). *Let p^* denote the plane which is the harmonic conjugate of the tangent plane $z=0$ with respect to the planes p_1, p_2. Then, the three planes*

p^*_i $(i = 1, 2, 3)$ corresponding to the Segre tangents at P_x are concurrent in the canonical line $l_1(k)$ for which k is given by the formula (3.2).

4. The case when $L = -(2/3)[(1-h)/(1-3h)]$. In this section we assume that L is defined by equation (2.11). Equation (2.3) becomes, in consequence of (2.6) and (2.11),

$$xy - z + h(\gamma\lambda^2 xz + \beta yz/\lambda^2) - \left[\frac{2}{3}\left(\frac{1-h}{1-3h}\right)\beta\gamma + \tfrac{1}{2}\theta_{uv}\right] z^2 = 0, \tag{4.1}$$

where h is a constant which is independent of u and v. Thus, *the conjugal quadrics* (4.1) *are characterized by the property that the envelope of the plane* (2.5) *is different from the cone* (2.10)

It may be remarked that the quadric (4.1) with $h = 0$ is the quadric of Darboux which Chang introduced [4; p. 928].

We now proceed to find the envelope of the plane π'_λ for the quadrics represented by equation (4.1). For this purpose we replace equations (2.8) by the equations

$$\begin{aligned} &[(1-3h)\gamma\omega u_3 - 2\gamma\pi u_2]\lambda^3 + (1-3h)\gamma p u_3\lambda^2 \\ &\qquad + 2\beta\pi u_3\lambda + (1-3h)\beta\pi u_2 = 0, \\ &\pi\lambda^2 + \omega\lambda + p = 0, \end{aligned} \tag{4.2}$$

in which ω, π, and p are given by

$$\begin{aligned} \omega &= \tfrac{1}{2}(7+3h)\psi u_2 + \tfrac{1}{2}(7+3h)\phi u_3 - 6(1+3h)(1-h)u_4, \\ \pi &= -\frac{2(1+3h)(1-h)}{(1-3h)}\gamma u_2 + (1-6h)\psi u_3, \\ p &= (1-6h)\phi u_2 - \frac{2(1+3h)(1-h)}{(1-3h)}\beta u_3. \end{aligned} \tag{4.3}$$

Elimination of λ from equations (4.2) shows that *the envelope of the plane π'_λ is a cone of the sixth class whose equations in plane coördinates are $u_1 = 0$ and*

(4.4)

$$\begin{vmatrix} (1-3h)\gamma p u_3 & 2\gamma p u_2 & (1-3h)\beta\pi u_2 & (1-3h)\beta\omega u_2 - 2\beta p u_3 \\ (1-3h)\gamma\omega u_3 - 2\gamma\pi u_2 & (1-3h)\gamma p u_3 & 2\beta\pi u_3 & (1-3h)\beta\pi u_2 \\ \pi & \omega & p & 0 \\ 0 & \pi & \omega & p \end{vmatrix} = 0.$$

The second polar of the tangent plane $z = 0$ with respect to the cone (4.4) has the equations $u_1 = 0$ and

$$(4.5)\qquad (1-3h)\pi\omega u_2^2+(1-3h)p\omega u_3^2 - (1-3h)^2\omega^2u_2u_3+2h(3h-2)\pi pu_2u_3=0.$$

If $h=0$, the second polar (4.5) decomposes into a cone of the third class and a pencil of planes whose axis is the canonical line $l_1(k)$ for which $k=-7/12$, as Chang has shown. Moreover, if $h=2/3$, the second polar also decomposes into a cone of the third class and a pencil of planes whose axis is the canonical line $l_1(k)$ for which $k=-3/4$. We summarize these results in the following statement:

Among the quadrics (4.1), *there are two quadrics, namely, those with* $h=0$ *and* $h=2/3$, *which are characterized by the property that the corresponding cone* (4.5) *decomposes into a cone of the third class and a pencil of planes. The axes of these two pencils of planes are the canonical lines* $l_1(k)$ *for which* k *has the respective values* $k=-7/12$ *and* $k=-3/4$.

If we exclude the two values of h for which the cone (4.5) is composite, it is easy to show that *the tangent planes of the cone* (4.5) *which pass through the asymptotic tangents at the point* P_x *intersect in the canonical line* $l_1(k)$ *for which* k *is given by the formula* (3.7).

The third polar of the tangent plane $z=0$ with respect to the cone (4.4) is the cone (3.1). Finally, the fourth polar of the tangent plane $z=0$ with respect to the cone (4.4) consists of two pencils of planes each having the asymptotic tangents at P_x for an axis.

5. The case when $h=-1/3$. It has been remarked that equation (2.3) with $h=-1/3$ defines a pencil of quadrics each of which has third order contact with the curves of the net N_λ at P_x.

In this case the homogeneous plane coördinates of the plane (2.5) are

$$(5.1)\qquad \begin{aligned}\rho u_1&=0,\\ \rho u_2&=24\lambda^2(\beta-\gamma\lambda^3),\\ \rho u_3&=-24\lambda(\beta-\gamma\lambda^3),\\ \rho u_4&=4\beta^2-3\beta\phi\lambda+12\beta\psi\lambda^2+(8\beta\gamma+36L\beta\gamma)\lambda^3+12\gamma\phi\lambda^4\\ &\quad -3\gamma\psi\lambda^5+4\gamma^2\lambda^6.\end{aligned}$$

Homogeneous elimination of λ from these equations yields *a cone of the sixth class whose equations are*

$$(5.2)\qquad \begin{aligned}&4(\beta u_3^3+\gamma u_2^3)(\beta u_3^3+\gamma u_2^3+\tfrac{3}{4}\phi u_2u_3^2+\tfrac{3}{4}\psi u_2^2u_3-6u_2u_3u_4)\\ &+9\beta\psi u_2^2u_3^4+9\gamma\phi u_2^4u_3^2-(16\beta\gamma+36L\beta\gamma)u_2^3u_3^3=0,\quad u_1=0.\end{aligned}$$

The following statement may be easily verified.

The first polar of planes through the asymptotic u-tangent (v-tangent) at the point P_x with respect to the cone (5.2) contains a plane $p_1(p_2)$ which passes through the v-tangent (u-tangent) at P_x. The intersection of the planes p_1 and p_2 is the canonical line $l_1(k)$ for which $k = -1/8$.

Moreover, the first polar of planes through the u-tangent at P_x with respect to the cone (5.2) contains a plane through each of the three tangents of Segre at P_x. The equations of these planes π_i $(i = 1, 2, 3)$ are

$$(5.3)\qquad \begin{aligned} &24(4\beta - \gamma\lambda^3)(\lambda x - y) + [24\beta^2/\lambda - 15\beta\phi \\ &\quad + 48\beta\psi\lambda + 3(8\beta\gamma + 36L\beta\gamma)\lambda^2 + 24\gamma\phi\lambda^3 - 3\gamma\psi\lambda^4]z = 0, \end{aligned}$$

where

$$\lambda = \omega^i(\beta/\gamma)^{1/3}; \quad i = 1, 2, 3; \quad \omega^3 = 1, \ \omega \neq 1.$$

The polar line of the tangent plane $z = 0$ with respect to the trihedron formed by the planes π_i is the line l_1 which joins P_x to the point whose local point coördinates are

$$(5.4)\qquad (0, -\tfrac{5}{8}\psi, \tfrac{1}{8}\phi, 1).$$

In a similar manner, we find that the first polar of planes through the v-tangent with respect to the cone (5.2) contains a plane through each of the tangents of Segre at P_x. These three planes π^*_i $(i = 1, 2, 3)$ are given by the equations

$$(5.5)\qquad \begin{aligned} &24(4\gamma\lambda^3 - \beta)(\lambda x - y) + [3\beta\phi - 24\beta\psi\lambda - 3(8\beta\gamma \\ &\quad + 36L\beta\gamma)\lambda^2 - 48\gamma\phi\lambda^3 + 15\gamma\psi\lambda^4 - 24\gamma^2\lambda^5]z = 0, \end{aligned}$$

where

$$\lambda = \omega^i(\beta/\gamma)^{1/3}; \quad i = 1, 2, 3; \quad \omega^3 = 1, \ \omega \neq 1.$$

The polar line of the tangent plane $z = 0$ with respect to the trihedron formed by the planes π^*_i is the line l^*_1 joining P_x to the point

$$(5.6)\qquad (0, \tfrac{1}{8}\psi, -\tfrac{5}{8}\phi, 1).$$

The plane determined by the lines l_1 and l^*_1 thus defined intersects the canonical plane in the first edge of Green.

We present a geometric characterization of a unique conjugal quadric (2.3) with $h = -1/3$ which is contained in the following theorem:

The third polar of planes through the u-tangent (v-tangent) with respect to the cone (5.2) *contains a plane which passes through the v-tangent (u-tangent) if, and only if,* $L = -2/9$. *In this case the quadric has the equation*

$$(5.7) \qquad xy - z - \tfrac{1}{3}(\gamma\lambda^2 xz + \beta yz/\lambda^2) - (\tfrac{2}{9}\beta\gamma + \tfrac{1}{2}\theta_{uv})z^2 = 0.$$

Finally, direct calculation, which will be omitted, yields the following result:

At a point P_x *of a surface, the fourth polar of planes through the u-tangent (v-tangent) with respect to the cone* (5.2) *is independent of* L *and contains a plane which passes through the v-tangent (u-tangent). The line of intersection of these two planes is the first directrix of Wilczynski.*

6. The case when $h = 1/3$. The quadric (2.3) with $h = 1/3$ is related to the associate conjugate net of the net N_λ in the same way that the quadric with $h = -1/3$ is related to the net N_λ.

The plane coördinates of the plane (2.5) are

$$(6.1) \qquad \begin{aligned} &\rho u_1 = 0, \qquad \rho u_2 = 24\beta\lambda^2, \qquad \rho u_3 = 24\gamma\lambda^4, \\ &\rho u_4 = 4\beta^2 - 3\beta\phi\lambda + 12\beta\psi\lambda^2 + (8\beta\gamma + 36L\beta\gamma)\lambda^3 + 12\gamma\phi\lambda^4 \\ &\qquad - 3\gamma\psi\lambda^5 + 4\gamma^2\lambda^6. \end{aligned}$$

The envelope of this plane when λ varies is found to be *a cone of the sixth class whose equations are*

$$(6.2) \qquad \begin{aligned} &16\beta\gamma[\beta u_3^3 + \gamma u_2^3 - 3(2u_4 - \phi u_3 - \psi u_2)u_2u_3]^2 \\ &\quad - [3(\beta\psi u_3^3 + \gamma\phi u_2^2) - (8\beta\gamma + 36L\beta\gamma)u_2u_3]^2 u_2u_3 = 0, \quad u_1 = 0. \end{aligned}$$

The first polar of the tangent plane $z = 0$ with respect to this cone decomposes into a cone of the third class

$$(6.3) \qquad \beta u_3^3 + \gamma u_2^3 - 3(2u_4 - \phi u_3 - \psi u_2)u_2u_3 = 0, \quad u_1 = 0,$$

and two pencils of planes each having an asymptotic tangent as axis. It will be observed that *the cusp-axis of the cone* (6.3) *is the first directrix of Wilczynski.*

There are two planes p_1, p_2 through the tangent line in the direction λ

at the point P_x which are tangent to the cone (6.2). The harmonic conjugate of the tangent plane $z=0$ with respect to the planes p_1, p_2 is the plane p^* whose equation is

$$6\lambda(x_3-\lambda x_2)+[3(\phi-\psi\lambda)\lambda-(\beta-\gamma\lambda^3)]x_4=0. \tag{6.4}$$

The planes p^*_i $(i=1,2,3)$ corresponding to the three tangents of Segre are concurrent in the first directrix of Wilczynski.*

7. The case when $h=1$. The quadrics of Davis for the net N_λ are the quadrics (2.3) with $h=1$. In this case the plane coördinates of the plane (2.5) are

$$\begin{aligned}&\rho u_1=0, \qquad \rho u_2=24\lambda^2(\beta+\gamma\lambda^3), \qquad \rho u_3=24\lambda(\beta+\gamma\lambda^3),\\ &\rho u_4=4\beta^2-3\beta\phi\lambda+12\beta\psi\lambda^2+(8\beta\gamma+36L\beta\gamma)\lambda^3+12\gamma\phi\lambda^4\\ &\qquad -3\gamma\psi\lambda^5+4\gamma^2\lambda^6.\end{aligned} \tag{7.1}$$

The envelope of this plane is *a cone of the sixth class with the equations*

$$\begin{aligned}&4(\beta u_3^3+\gamma u_2^3)(\beta u_3^3+\gamma u_2^3-\tfrac{3}{4}\phi u_2u_3^2-\tfrac{3}{4}\psi u_2^2u_3-6u_2u_3u_4)\\ &\quad+15\beta\psi u_2^2u_3^4+15\gamma\phi u_2^4u_3^2+36L\beta\gamma u_2^3u_3^3=0, \quad u_1=0.\end{aligned} \tag{7.2}$$

It is not difficult to show that the first polar of planes through the u-tangent at P_x with respect to the cone (7.2) contains a plane through each of the tangents of Segre at P_x. The polar line of the tangent plane $z=0$ with respect to the trihedron formed by these three planes is the line l_1 which joins P_x to the point

$$(0, \ -{}^{17}\!/_{24}\psi, \ {}^{13}\!/_{24}\phi, \ 1). \tag{7.3}$$

Similarly, the first polar of planes through the v-tangent with respect to the cone (7.2) contains a plane through each of the Segre tangents. The polar line of the tangent plane $z=0$ with respect to this trihedron is the line l^*_1 which joins P_x to the point

$$(0, \ {}^{13}\!/_{24}\psi, \ -{}^{17}\!/_{24}\phi, \ 1). \tag{7.4}$$

The plane determined by the lines l_1, l^*_1 intersects the canonical plane in the canonical line $l_1(k)$ for which $k=-1/12$.

Finally, direct calculation yields the following result:

The third polar of planes through the u-tangent (v-tangent) with

respect to the cone (7.2) *contains a plane which passes through the v-tangent* (*u-tangent*) *if, and only if,* $L = -2/9$. *Consequently, we have a geometric characterization of the unique quadric represented by the equation*

$$(7.5) \qquad xy - z + \gamma\lambda^2 xz + \beta yz/\lambda^2 - (\tfrac{2}{9}\beta\gamma + \tfrac{1}{2}\theta_{uv})z^2 = 0.$$

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THE INFINITIES IN THE NON-LOCAL EXISTENCE PROBLEM OF ORDINARY DIFFERENTIAL EQUATIONS.*

By AUREL WINTNER.

The standard general existence theorem for the problem of initial values assigned for systems of ordinary differential equations is strictly local in nature. There seem to be just two standard types which are particular enough to allow a control of the solutions in the large. The first of these classical cases is represented by systems of linear differential equations, the second by those systems the solutions of which depend on inversions of quadratures.

This latter type is exemplified, in the complex domain, by the differential equation $dp/dz = (p^3 + ap + b)^{\frac{1}{2}}$ of the elliptic function $p(z)$ and, in the real domain, by Liouville's separable conservative systems with n degrees of freedom. In fact, if x denotes any of the n separating coordinates, the energy relation of each degree of freedom is a differential equation of the form $dx/dt = f(x)$. As illustrated by the example $f(x) = x^{1/3}$, where $-\infty < x < \infty$, the mere continuity of $f(x)$ is insufficient for the uniqueness of the solution $x(t)$ belonging to a given initial value $x(0)$, since $x(t) = \alpha \mid t \mid^{3/2} \operatorname{sgn} t$ then is a solution of $dx/dt = f(x)$, $x(0) = 0$, for *two* non-vanishing values of α (and also for $\alpha = 0$).

However, since the zeros of a continuous function form a closed set (and possess, therefore, a complement consisting of a sequence of mutually disjoint open intervals), it is easily realized that, no matter how complicated the real, continuous function $f(x)$, $-\infty < x < \infty$, may be, *a solution $x(t)$ of $dx/dt = f$ cannot cease to exist at a finite $t = t_0$ without tending either to $+\infty$ or to $-\infty$ as $t \to t_0$.* (That such a value t_0 can actually occur, is shown by the movable singularity of the solutions $x(t) = (t_0 - t)^{-1}$ of the differential equation $dx/dt = x^2$, in which $f(x)$ is regular throughout.) In the elliptic case of the complex field, the italicized remark manifests itself in the fact that the only points z ($\neq \infty$) at which $w = p(z + \text{const.})$ fails to satisfy its (not rational, though algebraic) differential equation $dw/dz = f(w)$ are infinities proper, since all of its singularities are poles, and so, in particular, neither of the type $(z - z_0)^{\frac{1}{2}}$ nor of the type $\exp(z - z_0)^{-1}$. And all singu-

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larities of $w(z)$ (for $z \neq \infty$) remain infinities proper in all hyperelliptic cases of $dw/dz = f(w)$. In this regard, it is besides the point that there are no branch points at all, not even infinities proper of the type $(z - z_0)^{-\frac{1}{2}}$, in the unramified case. With regard to branch points of infinite order, it is understood that a logarithmic singularity represents an infinity proper, if it is of the type $\log(z - z_0)$, but does not, if it is of the type $(z - z_0)\log(z - z_0)$ or $1/\log(z - z_0)$.

What concerns the first of the two standard cases mentioned at the very beginning, it was shown in [3] that, by a repeated application of the local existence theorem, it is possible to delimit a class of non-linear systems for which the behavior of all solutions in the large is typified by the classical case of linear systems. In the present note, there will be delimited a class of differential equations for which the other case, that depending on inversions of quadratures, is the prototype. By this is implied that, in the real domain, the property italicized above will hold for all solutions. In the complex domain, this must be replaced by the property that all (finite) singularities of all solutions are infinities proper in the sense described above. The principal device will consist of an adaptation of that application of the local existence theorem of arbitrary ordinary differential equations which was used by Painlevé both in the complex field (cf., e. g., [2], p. 23) and in his discussion of the real singularities of the problem of three bodies (*ibid.*, pp. 584-586).

In the real domain, the theorem to be proved becomes the following fact:

If $f(t, x)$ is a real-valued, continuous function on a strip

$$0 \leqq t \leqq a, \qquad -\infty < x < \infty,$$

and if a real-valued function

$$x = x(t), \qquad 0 \leqq t < t_0,$$

where $t_0 \leqq a$ (and $t_0 \neq \infty$), is a solution of the differential equation

$$x' = f(t, x),$$

then, as t tends from the left to the excluded value t_0,

either there exists a finite limit $x(t_0 - 0)$, in which case the function $x(t)$ can be extended to the closed interval $0 \leqq t \leqq t_0$ (and, if $t_0 \neq a$, to an interval $0 \leqq t \leqq t^*$, where $t_0 < t^* < a$) in such a way as to possess a continuous derivative and to satisfy the differential equation,

or else $|x(t)|$ *tends to* ∞, *which, since* $x(t)$ *is continuous for* $0 \leqq t < t_0$, *means that the limit* $x(t_0 - 0)$ *exists either as* $+\infty$ *or as* $-\infty$.

In particular, $\liminf |x(t)| < \infty$ is impossible unless lim-sup $|x(t)| < \infty$, as $t \to t_0 - 0$. Actually, what the theorem really claims is that $\liminf |x(t)| < \infty$ is impossible unless there exists a (finite) $\lim x(t)$, as $t \to t_0 - 0$. In fact, it is easy to see from well-known general properties of functions which are derivatives (and then, if $t_0 \neq a$, from the local existence theorem of ordinary differential equations), that this particular claim, though just a corollary, happens to be equivalent to the theorem.

Correspondingly, the content of the theorem can be illustrated as follows: No matter what the continuous function $f(t, x)$ (*on a strip*) may be, a solution $x(t)$ cannot behave as

$$\sin (t - t_0)^{-1} \quad \text{or} \quad (t - t_0)^{-1} \sin (t - t_0)^{-1},$$

since both of these functions satisfy $\liminf |x(t)| < \infty$ but possess no $\lim x(t)$, as $t \to t_0 - 0$. In addition, the behavior of

$$(t - t_0) \sin (t - t_0)^{-1} \quad \text{or} \quad (t - t_0)^2 \sin (t - t_0)^{-1}$$

is excluded, since, though both of these functions are continuous (if defined to be 0 at $t = t_0$), the first of them has no derivative, and the second no continuous derivative, at $t = t_0$.

It is easy to realize that generalizations for the case of an order higher than the first are not possible in certain directions. For instance, one of the simplest cases not covered by the theorem is a system of $n = 2$ differential equations of the first order not containing the independent variable, say

$$dx/dt = f(x, y), \qquad dy/dt = g(x, y),$$

where both functions f, g are regular on the whole (x, y)-plane. But the assertions of the theorem may then become false, even if both functions are restricted to be polynomials. This is proved by the example (cf. [2], p. 545)

$$f = y(x^2 + y^2)^2 + \tfrac{1}{2}x(x^2 + y^2), \qquad g = x(x^2 + y^2)^2 - \tfrac{1}{2}y(x^2 + y^2),$$

since, as is seen by differentiations and substitutions,

$$x(t) = (t_0 - t)^{-\frac{1}{2}} \sin (t_0 - t)^{-1}, \qquad y(t) = (t_0 - t)^{-\frac{1}{2}} \cos (t_0 - t)^{-1}$$

is then a (real) solution (for $-\infty < t < t_0$).

Let $x(t)$, where $0 \leqq t < t_0$, be a solution satisfying $\liminf |x(t)| < \infty$ as $t \to t_0 - 0$.. This means that there exist an increasing sequence $t_1, t_2, \cdots$ and a constant c satisfying $t_m \to t_0$ and $|x(t_m)| \leqq c$. Corresponding to this c and to an arbitrary positive constant b, choose an $M = M(b, c)$ so large that the inequality $|f(t, x)| \leqq M$ holds at every point of the rectangle

$$0 \leqq t \leqq t, \qquad -b-c \leqq x \leqq b+c.$$

The existence of such an M is assured by the assumption that $f(t, x)$ is given as a continuous function on the strip $0 \leqq t \leqq a$ of the (t, x)-plane.

The rectangle defined by the last formula line contains the rectangle

$$t_m \leqq t \leqq a, \qquad -b \leqq x - x(t_m) \leqq b$$

for every m, since $0 \leqq t_m < t_0 \leqq a$ and $|x(t_m)| \leqq c$. Hence, $|f(t, x)| \leqq M$ holds at every point of the rectangle just mentioned, where m is arbitrary. It follows therefore from the local existence theorem of ordinary differential equations, that $x' = f(t, x)$ has on the closed t-interval

$$t_m \leqq t \leqq t_m + \min(a - t_m, b/M)$$

a solution $x(t)$ attaining the given initial value $x(t_m)$ at $t = t_m$. Accordingly, in order to complete the proof of the theorem, it is sufficient to ascertain that, if m is large enough, the upper end-point, $t = t_m + \min$, of this closed t-interval is beyond or at the point $t = t_0$ according as $t_0 < a$ or $t_0 = a$. But this is implied by the assumption $t_m \to t_0$, since b/M is independent of m.

Appendix.

By an adaptation of the proof given above, it is possible to obtain, in a generalized form, a simple proof of the principal result of [3]. The generalization consists in omitting an assumption of monotony for the function $L(r)$ below. The resulting theorem, which is a complete dual of Osgood's criterion for the uniqueness of the solutions, is as follows:

Let $f_1, \cdots, f_n$ be real-valued, continuous functions on the $(n+1)$-dimensional region

$$\text{(i)} \qquad 0 \leqq t \leqq a, \qquad -\infty < x_i < \infty \qquad (i = 1, \cdots, n).$$

Suppose that there exists a continuous function $L(r)$, $0 \leqq r < \infty$, satisfying on the region (i) *the n inequalities*

(ii) $$|f_i(t; x_1, \cdots, x_n)| < L(r),$$

where

(iii) $$r^2 = x_1^2 + \cdots + x_n^2,$$

and having the property that

(iv) $$\int^{\infty} dr/L(r) = \infty.$$

Then, if $c_1, \cdots, c_n$ is any set of integration constants, the differential equations

(v) $$x_i' = f_i(t; x_1, \cdots, x_n)$$

and the initial conditions

(vi) $$x_i(0) = c_i$$

have a solution $x_i = x_i(t)$ on the whole t-range, $0 \leqq t \leqq a$, admitted in (i).

Suppose that this assertion is false. Then there exists a positive number t_0 not exceeding the value of a and having the property that (v) and (vi) have on the interval $0 \leqq t < t_0$ a solution $x_i = x_i(t)$ which cannot be extended to the closed interval $0 \leqq t \leqq t_0$. Hence, if the local existence theorem of (v) and (vi) is applied in exactly the same way as above (where $n = 1$), it follows that there cannot exist an increasing sequence $t_1, t_2, \cdots$ tending to t_0 and having the property that each of the n sequences $x_i(t_1), x_i(t_2), \cdots$ is bounded. But the non-existence of such a sequence $t_1, t_2, \cdots$ means, by (iii), that

(vii) $$r = r(t) \to \infty \quad \text{as} \quad t \to t_0 - 0.$$

And it is easy to see that (vii) contradicts (ii) and (iv).

In fact, differentiation of (iii), where $r = r(t)$, gives

$$(rr')^2 \leqq (x_1^2 + \cdots + x_n^2)(x_1'^2 + \cdots + x_n'^2) = r^2(f_1^2 + \cdots + f_n^2),$$

by (v). Hence, from (ii),

$$dt \geqq |dr|/L(r),$$

if the constant factor $n^{1/2}$ is thought of as absorbed by the function sign L. But the last formula line implies that the inequality

$$\lim_{t \to t_0 - 0} t - c \geqq \lim_{R \to \infty} \int_C^R dr/L(r)$$

holds for a certain pair of integration constants c, C. Since the limit on the left has a finite value, this inequality contradicts (iv).

It is clear from the proof that the continuity of the positive function $L(r)$ can be replaced by more general local assumptions.

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REDUCTION OF AN n-TH ORDER LINEAR DIFFERENTIAL EQUATION AND m-POINT BOUNDARY CONDITIONS TO AN EQUIVALENT MATRIX SYSTEM.*

By RANDAL H. COLE.

1. Introduction. The linear differential equation of order n,

$$u^{(n)} + P_1 u^{(n-1)} + \cdots + P_n u = 0, \tag{1.1}$$

with boundary conditions of various types has been extensively discussed by Birkhoff [1],[1] Wilder [2], Tamarkin [3], and others. Much attention has also been devoted to systems of first order equations,[2] and in particular to systems which may be written in the matrix[3] form,

$$\mathfrak{Y}' = \{\lambda\mathfrak{R} + \mathfrak{Q}\}\mathfrak{Y}. \tag{1.2}$$

Here, as before, the problems are largely characterized by the type of the adjoined boundary relations. The results obtained by Birkhoff and Langer [5] in 1923 and by Langer [6] in 1939 may be cited as examples of significant developments in this field.

Although (1.1) is readily reducible to a system of n first order linear equations, obvious reductions do not yield systems which are linear in the parameter. A reduction of (1.1) to (1.2) has been obtained by Wilder [7], but it is not carried to the point where the matrices $\mathfrak{R}$ and $\mathfrak{Q}$ are in a form which permits the immediate application of known results for expansion problems. Further, the reducibility of boundary conditions associated with (1.1) to an equivalent set associated with (1.2) has not been considered.

The present paper shows that (1.1) is reducible to a familiar and convenient form of system (1.2), and that a set of m-point boundary conditions applying to (1.1) admits of a corresponding reduction to a set of matrix boundary conditions applying to (1.2). The theory of differential matrix

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[1] Numbers in brackets refer to the bibliography.

[2] A review of the literature associated with such systems has been given by W. M. Whyburn [4].

[3] Square matrices of order n will be designated by German capital letters and their components will be represented by the corresponding lower case italic letters, that is, $\mathfrak{B}(x) \equiv (b_{i,j}(x))$. Similarly the lower case German letters will be reserved exclusively for the representation of vectors.

equations is, therefore, made available for the extension and generalization of the classical boundary and expansion problems associated with the n-th order linear differential equation.

2. Reduction of the n-th order equation to matrix form. The equation to be considered is (1.1) for which it is assumed that

(a) *each coefficient P_k is a polynomial in λ of the form*

$$P_k = \sum_{l=0}^{k} P_{k,l}(x)\lambda^l,$$

with $P_{k,l}(x)$ free from λ and indefinitely differentiable,

(b) *the algebraic equation,*

$$r^n + P_{1,1}r^{n-1} + \cdots + P_{n,n} = 0,$$

has n distinct roots, $r_1(x), r_2(x), \cdots, r_n(x)$, for all values of x on a fundamental interval (a, b).

Under these assumptions there are known to exist [7] two indefinitely differentiable matrices, $\mathfrak{B}(x)$ and $\mathfrak{C}(x)$, free from λ and such that any matrix solution of

$$\mathfrak{Z}' = \{\lambda\mathfrak{B}(x) + \mathfrak{C}(x)\}\mathfrak{Z} \tag{2.1}$$

has solutions of (1.1) as the components of its first row. Further, $\mathfrak{B}(x)$ and $\mathfrak{C}(x)$ are such that

$$b_{i,j}(x) = 0, \qquad j > i,$$
$$b_{i,i}(x) = r_i(x),$$

and

$$c_{i,j}(x) = 0, \qquad j > i+1.$$
$$c_{i,i+1}(x) = 1.$$

A matrix of the form of $\mathfrak{B}(x)$ will henceforth be referred to as a lower triangular matrix and one of the form of $\mathfrak{C}(x)$ as a bordered lower triangular matrix.

In view of the form of $\mathfrak{B}(x)$ there exists a non-singular matrix $\mathfrak{T}(x)$ such that

$$\mathfrak{T}^{-1}(x)\mathfrak{B}(x)\mathfrak{T}(x) = \mathfrak{R}(x),$$

where [4]

$$\mathfrak{R}(x) = (\delta_{i,j} r_j(x)).$$

Since $\mathfrak{T}$ satisfies the relation $\mathfrak{B}\mathfrak{T} = \mathfrak{T}\mathfrak{R}$, it may be chosen of lower triangular form with 1's on the main diagonal. The matrix $\mathfrak{T}^{-1}$ will then be similarly constituted. With this choice of $\mathfrak{T}$ it may be verified that $\mathfrak{Q}(x)$, defined by

$$\mathfrak{Q}(x) = \mathfrak{T}^{-1}(x)\mathfrak{C}(x)\mathfrak{T}(x) - \mathfrak{T}^{-1}(x)\mathfrak{T}'(x),$$

is a bordered lower triangular matrix. For $\mathfrak{R}$ and $\mathfrak{Q}$ so defined, a solution of (2.1) yields, through the relation $\mathfrak{Z}(x, \lambda) = \mathfrak{T}(x)\mathfrak{Y}(x, \lambda)$, a solution of

$$\mathfrak{Y}' = \{\lambda\mathfrak{R}(x) + \mathfrak{Q}(x)\}\mathfrak{Y}. \tag{2.2}$$

In view of the form of $\mathfrak{T}$, it follows that any matrix solution of (2.2) will have solutions of (1.1) as components of its first row. Further, if any solution of (2.2) is non-singular, the components of its first row constitute a fundamental set of solutions of (1.1). This fact may be established by considering the relations, defined by (2.2), which the components of $\mathfrak{Y}$ must satisfy. The form of $\mathfrak{R}$ and $\mathfrak{Q}$ is such that these relations reduce to

$$y'_{i,j} = \lambda r_i y_{i,j} + \sum_{\nu=1}^{i} q_{i,\nu} y_{\nu,j} + y_{i+1,j}.$$

It is, therefore, clear that if the first i components of the j-th column of $\mathfrak{Y}$ are identically zero, the $(i+1)$-th component is also identically zero. Thus, if the first component of the j-th column is zero, all the components of that column are zero and $\mathfrak{Y}$ is singular. This implies that if $\mathfrak{Y}$ is non-singular, no component of its first row is zero. If $\mathfrak{C}$ is any non-singular matrix free from x, $\mathfrak{Y}\mathfrak{C}$ is also a non-singular solution and, by the same argument, has no zero components in its first row. Hence, if the components of the first row of $\mathfrak{Y}$ were not linearly independent, an appropriate choice of $\mathfrak{C}$ would furnish a contradiction.

We have shown that the first row of $\mathfrak{Y}$ is made up of u's, but there is still the question of the relationship between the derivatives of these u's and the components of $\mathfrak{Y}$. With this question in view, let the square matrix $\mathfrak{N}(u_1, u_2, \cdots, u_n)$, associated with any n functions $u_1(x), u_2(x), \cdots, u_n(x)$, each possessing $(n-1)$ derivatives, be defined by

$$\mathfrak{N}(u_1, u_2, \cdots, u_n) = (u_j^{(i-1)}(x)).$$

[4] The symbol $\delta_{i,j}$ is used in the sense, $\delta_{i,j} = 0$, if $i \neq j$; $\delta_{i,j} = 1$, if $i = j$.

We shall show that there exists a non-singular matrix $\mathfrak{S}(x,\lambda)$, such that, if $\mathfrak{Y}(x,\lambda)$ is any matrix solution of (2.2),

$$\mathfrak{S}(x,\lambda)\mathfrak{Y}(x,\lambda) = \mathfrak{K}(y_{1,1},\cdots,y_{1,n}), \tag{2.3}$$

where $y_{1,1},\cdots,y_{1,n}$, being components of the first row of $\mathfrak{Y}(x,\lambda)$, are solutions of (1.1).

To derive $\mathfrak{S}$, let the matrix $\mathfrak{S}^{(\eta)}$ be defined by the recurrence relation

$$\mathfrak{S}^{(\eta)} = d\mathfrak{S}^{(\eta-1)}/dx + \mathfrak{S}^{(\eta-1)}\mathfrak{S}^{(1)},$$

with $\mathfrak{S}^{(0)} = (\delta_{i,j})$ and $\mathfrak{S}^{(1)} = \{\lambda\mathfrak{R}(x) + \mathfrak{Q}(x)\}$. The η-th derivative of any matrix solution $\mathfrak{Y}$ is, therefore, given by

$$\mathfrak{Y}^{(\eta)} = \mathfrak{S}^{(\eta)}\mathfrak{Y}, \qquad (\eta = 0, 1, 2, \cdots).$$

The first row of $\mathfrak{Y}^{(\eta-1)}$ is clearly the η-th row of $\mathfrak{K}(y_{1,1},\cdots,y_{1,n})$ so that

$$\mathfrak{K}(y_{1,1},\cdots,y_{1,n}) = \sum_{\eta=1}^{n} (\delta_{i,\eta}\delta_{1,j})\mathfrak{S}^{(\eta-1)}\mathfrak{Y}.$$

The relation (2.3) follows immediately with $\mathfrak{S}(x,\lambda)$ defined by

$$\mathfrak{S}(x,\lambda) = \sum_{\eta=1}^{n} (\delta_{i,\eta}\delta_{1,j})\mathfrak{S}^{(\eta-1)}. \tag{2.4}$$

When so defined, $\mathfrak{S}(x,\lambda)$ is readily seen to have components which are polynomials in λ. We may show that it is non-singular by assuming that the components of $\mathfrak{S}^{(\eta)}$ are such that

$$s_{i,j}{}^{(\eta)} \equiv 0 \text{ if } j > i+\eta; \qquad s_{i,i+\eta}^{(\eta)} \equiv 1.$$

The component in the i-th row and j-th column of $\mathfrak{S}^{(\eta+1)}$ is therefore given by

$$s_{i,j}{}^{(\eta+1)} = ds_{i,j}{}^{(\eta)}/dx + \sum_{\nu=j-1}^{i+\eta} s_{i,\nu}{}^{(\eta)} s_{\nu,j}{}^{(1)},$$

which implies that

$$s_{i,j}{}^{(\eta+1)} \equiv 0 \text{ if } j > i+\eta+1; \qquad s_{i,i+\eta+1}^{(\eta+1)} \equiv 1.$$

That is, the matrix $\mathfrak{S}^{(\eta+1)}$ is also of the assumed form and, since $\mathfrak{S}^{(0)}$ and $\mathfrak{S}^{(1)}$ are known to be of this form, the validity of the assumption is established by induction. In view of (2.4), therefore, $\mathfrak{S}(x,\lambda)$ is a lower triangular matrix with unit components on the main diagonal and is non-singular.

3. Boundary conditions. The boundary conditions to be adjoined to (1.1) apply at the end points of the fundamental interval and also at an

arbitrary set of intermediate points, $a_2, a_3, \cdots, a_{m-1}$. If the end points, a and b, are denoted by a_1 and a_m, respectively, and if the designation is such that $a_\mu < a_{\mu+1}$, $(\mu = 1, 2, \cdots, m-1)$, these conditions are given by

$$(3.1) \quad V_i^{(1)}(u) + V_i^{(2)}(u) + \cdots + V_i^{(m)}(u) = 0, \quad (i = 1, 2, \cdots, n),$$

where

$$V_i^{(\mu)}(u) = v_{i,1}^{(\mu)} u(a_\mu, \lambda) + v_{i,2}^{(\mu)} u'(a_\mu, \lambda) + \cdots + v_{i,n}^{(\mu)} u^{(n-1)}(a_\mu, \lambda)$$

with coefficients, $v_{i,j}^{(\mu)}$, analytic functions of λ. If $\mathfrak{k}(x, \lambda)$ is defined to be the vector whose components are $u(x, \lambda)$ and its first $(n-1)$ derivatives. these relations may be written in the matrix form,

$$\sum_{\mu=1}^{m} \mathfrak{V}^{(\mu)}(\lambda) \mathfrak{k}(a_\mu, \lambda) = 0,$$

with

$$\mathfrak{V}^{(\mu)}(\lambda) = (v_{i,j}^{(\mu)}(\lambda)).$$

Consider the system

$$(3.2\,a) \qquad \mathfrak{y}' = \{\lambda \mathfrak{R}(x) + \mathfrak{Q}(x)\}\mathfrak{y}$$

$$(3.2\,b) \qquad \sum_{\mu=1}^{m} \mathfrak{W}^{(\mu)}(\lambda) \mathfrak{y}(a_\mu, \lambda) = 0,$$

where $\mathfrak{W}^{(\mu)}(\lambda)$ is defined as a matrix of analytic functions of λ by

$$\mathfrak{W}^{(\mu)}(\lambda) = \mathfrak{V}^{(\mu)}(\lambda) \mathfrak{S}(a_\mu, \lambda).$$

The equation (3.2 a) is the vector equation corresponding to (2.2) and has the general solution $\mathfrak{Y}(x, \lambda)\mathfrak{c}(\lambda)$, where $\mathfrak{Y}(x, \lambda)$ is any non-singular solution of (2.2) and $\mathfrak{c}(\lambda)$ is an arbitrary vector free from x. The first component of any solution of (3.2) is clearly a solution of (1.1) and (3.1), and conversely, any solution of the latter system will yield, under appropriate transformations, a solution of (3.2).

The results obtained by Langer [6] for a system in the complex domain represent significant developments when specialized to the present problem. It is, therefore, worthy of note that the substitution (cf. [6, p. 154])

$$\mathfrak{y}(x, \lambda) = (\delta_{i,j} e^{\phi_j(x)}) \mathfrak{u}(x, \lambda),$$

where $\phi'_j(x) = q_{j,j}(x)$, reduces (3.2) to Langer's system. The substitution has the effect of replacing (3.2) by an equivalent and similar system in which

$\mathfrak{R}(x)$ is unchanged and $\mathfrak{Q}(x)$ is replaced by a matrix whose main diagonal components are identically zero.

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